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ORDER REDUCED METHODS FOR QUAD-CURL EQUATIONS WITH NAVIER TYPE BOUNDARY CONDITIONS*

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Abstract

Quad-curl equations with Navier type boundary conditions are studied in this paper. Stable order reduced formulations equivalent to the model problems are presented, and finite element discretizations are designed. Optimal convergence rates are proved.

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Key words: Quad-curl equation, Order reduced scheme, Regularity analysis, Finite element method.

1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be a contractable polyhedron domain, $\Gamma = \partial\Omega$ be the boundary of Ω, and $\mathbf{n}$ be the outer unit normal vector of $\partial\Omega$. As usual, we use ∇ for the gradient operator, and denote $\mathrm{div} = \nabla\cdot$ and $\mathrm{curl} = \nabla\times$. In this paper, we study the quad-curl problem of the type

$$(\mathbf{A}) \quad \begin{cases} (\nabla\times)^4 \underset{\sim}{u} = \underset{\sim}{f}, & \text{in } \Omega; \\ \nabla\cdot \underset{\sim}{u} = 0, & \text{in } \Omega; \\ \underset{\sim}{u} \times\mathbf{n} = (\nabla\times)^2 \underset{\sim}{u} \times\mathbf{n} = \underset{\sim}{0}, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\mathrm{div}\, \underset{\sim}{f} = 0$, and of the variant type

$$(\mathbf{B}) \quad \begin{cases} (\nabla\times)^4 \underset{\sim}{u} + \underset{\sim}{u} = \underset{\sim}{f}, & \text{in } \Omega; \\ \underset{\sim}{u} \times\mathbf{n} = (\nabla\times)^2 \underset{\sim}{u} \times\mathbf{n} = \underset{\sim}{0}, & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

For (1.2), it is not necessary that $\mathrm{div}\, \underset{\sim}{f} = 0$; however, when this is the case, then $\mathrm{div}\, \underset{\sim}{u} = 0$.

The quad-curl operator $(\nabla\times)^4$ arises in models for a wide variety of domains in applied sciences, including elasticity, inverse electromagnetic scattering theory, and magnetohydrodynamics (MHD). In elasticity, the operator is used to model the effect of the couple stress (c.f. [22, 28]). In inverse electromagnetic scattering theory (c.f. [9, 10]), $(\nabla\times)^4$ appears in the computation of the transmission eigenvalue. In MHD (c.f. [32]), $(\nabla\times)^4\mathbf{B}$ is involved in the

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resistive system, where $\mathbf{B}$ is the magnetic field as a primary variable. The quad-curl operator is also used as the principal part of the Electron MHD model; see Equation (1) of [11].

The mathematical modeling and numerical analysis of the quad-curl operator has been attracting interest from many researchers. The earlier results presented by the community focus on the Dirichlet-type boundary conditions of the types $\underset{\sim}{u}\times\mathbf{n} = (\nabla\times\underset{\sim}{u})\times\mathbf{n} = \underset{\sim}{0}$ and $\underset{\sim}{u}\times\mathbf{n} = \nabla\times\underset{\sim}{u} = \underset{\sim}{0}$. The fourth order operator $(\nabla\times)^4$ establishes a very large kernel space, which results in a complex intrinsic structure. More recently, Nicaise [21] proves that the solution of quad-curl equation does not always belong to H^3 on polyhedrons with H^{-1} data $\underset{\sim}{f}$, and singularities can be expected in general; Zhang [29] establishes the H^2 regularity under the assumptions of convex domains with L^2 input data.

The community has also explored alternative approaches for the discretization of the quad-curl problem. With a focus on the Dirichlet boundary conditions, both primal and reduced schemes have been proposed to address the complexity issue. Primal schemes include the proposals by Zheng-Hu-Xu [32] using nonconforming elements and by Hong-Hu-Shu-Xu [19] using standard high order Nédélec's elements within the framework of the discontinuous Galerkin method. Reduced schemes, also known as mixed element schemes, have received substantial attention. A mixed element scheme by Sun [25] suggests solving the original problem using existing edge elements. The scheme can be viewed as an analogue of the Ciarlet-Raviart's scheme [12] for the biharmonic equation in the context of the quad-curl problem; the equivalence to the primal formulation can be proved under additional assumptions on the regularity of the solution. Zhang [29] has proposed schemes based on the underlying de Rham complex structure for two variants of the quad-curl equations; their equivalence to the primal formulation can be proved without additional assumptions. The intrinsic topological characteristic, namely the norm used, is clearly presented and accompanied with the stability analysis. These results are used to design the optimal solvers/preconditioners for the scheme. It is worth remarking that, in [29], it is proved that the boundary value problems associated with the two Dirichlet types of boundary conditions are equivalent to each other. Very recently, the decomposition of the quad-curl problem in two dimensions has been presented by Brenner-Sun-Sung [8]; the multiply-connected characteristic of the domain is considered. Meanwhile, a coupled order reduced formulation for the quad-curl problem is also introduced in Zhang [30]. This work presents a general framework for reducing problems based on non-orthogonal decompositions; a specific example is presented for reducing quad-curl problems.

In contrast to the research that focuses on the Dirichlet boundary conditions, in this paper, we investigate the quad-curl equation with the Navier type boundary condition as in (1.1) and (1.2), and propose novel order reduced schemes. The motivation for this research is two-fold. First, the Navier boundary condition is important, as it can be found in various mechanical and physical systems. For example, according to the Maxwell system for a smooth electric field $\underset{\sim}{E}$, which satisfies a wave-form equation, it is natural to suppose $(\nabla\times)^2\underset{\sim}{E}\times\mathbf{n} = \underset{\sim}{0}$, provided $\underset{\sim}{E}\times\mathbf{n} = \underset{\sim}{0}$ is imposed as a boundary condition. Second, practically and theoretically, boundary conditions in general play a significant role in the study of the quad-curl equation; they provide a meaningful foundation from which to explore order reduced schemes.

Indeed, given the two kinds of Dirichlet type boundary conditions $\underset{\sim}{u}\times\mathbf{n} = (\nabla\times\underset{\sim}{u})\times\mathbf{n} = \underset{\sim}{0}$ and $\underset{\sim}{u}\times\mathbf{n} = \nabla\times\underset{\sim}{u} = \underset{\sim}{0}$ are equivalent on general polygons, the solution for this kind of boundary value problem $\underset{\sim}{u}$ can be expected for higher regularity, in particular for $\nabla\times\underset{\sim}{u}\in\underset{\sim}{H}^1(\Omega)$.