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SOLVING SYSTEMS OF QUADRATIC EQUATIONS VIA EXPONENTIAL-TYPE GRADIENT DESCENT ALGORITHM*

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Abstract

We consider the rank minimization problem from quadratic measurements, i.e., recovering a rank r matrix $X \in \mathbb{R}^{n\times r}$ from m scalar measurements $y_i = a_i^\top XX^\top a_i,\ a_i \in \mathbb{R}^n,\ i = 1,\ldots,m$. Such problem arises in a variety of applications such as quadratic regression and quantum state tomography. We present a novel algorithm, which is termed *exponential-type gradient descent algorithm*, to minimize a non-convex objective function $f(U) = \frac{1}{4m}\sum_{i=1}^m (y_i - a_i^\top UU^\top a_i)^2$. This algorithm starts with a careful initialization, and then refines this initial guess by iteratively applying exponential-type gradient descent. Particularly, we can obtain a good initial guess of X as long as the number of Gaussian random measurements is $O(nr)$, and our iteration algorithm can converge linearly to the true X (up to an orthogonal matrix) with $m = O\left(nr\log(cr)\right)$ Gaussian random measurements.

Mathematics subject classification: 90C26, 94A15.
Key words: Low-rank matrix recovery, Non-convex optimization, Phase retrieval.

1. Introduction

1.1. Problem setup.

Let $X \in \mathbb{R}^{n\times r}$ be a fixed and unknown matrix with $\text{rank}(X) = r$, and our aim is to recover X from given quadratic measurements, i.e.,

$$\text{find} \quad X \in \mathbb{R}^{n\times r}, \quad \text{s.t.} \quad y_i = a_i^\top XX^\top a_i = \|a_i^\top X\|_2^2, \qquad i = 1,\ldots,m, \tag{1.1}$$

where $a_i = (a_{i,1},\ldots,a_{i,n}) \in \mathbb{R}^n$. This problem is raised in many emerging applications of science and engineering, such as covariance sketching, quantum state tomography and high dimensional data streams [7,16,17]. A simple observation is that $a_i^\top XX^\top a_i = a_i^\top XOO^\top X^\top a_i$ where $O \in \mathbb{R}^{r\times r}$ is an orthogonal matrix. We can only hope to recover X up to a right orthogonal matrix. There exists an orthogonal matrix $O^* \in \mathbb{R}^{r\times r}$ such that XO^* has orthogonal column vectors. Hence, throughout the paper we can assume that X has orthogonal column vectors.

To recover X from given measurements (1.1), we consider the following optimization problem:

$$\min_{U\in\mathbb{R}^{n\times r}} f(U) = \frac{1}{4m}\sum_{i=1}^m (y_i - \|a_i^\top U\|_2^2)^2. \tag{1.2}$$

The aim of this paper is to develop algorithms to solve (1.2).

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1.2. Related work

1.2.1. Low rank matrix recovery

Rank minimization problem is a direct generalization of compressed sensing [15, 22]. For the general rank minimization problem, it aims to reconstruct a low rank matrix $Q \in \mathbb{R}^{n\times n}$ from incomplete measurements, which can be formulated as the following programming

$$\begin{aligned} \min_{Z\in\mathbb{R}^{n\times n}} \quad & \operatorname{rank}(Z) \\ \text{subject to} \quad & \operatorname{tr}(A_i Z) = y_i, \quad i = 1,\dots,m, \end{aligned} \tag{1.3}$$

where $y_i = \operatorname{tr}(A_i Q), A_i \in \mathbb{R}^{n\times n}, i = 1,\dots,m$. In [26], Xu has proved that in order to guarantee the solution of (1.3) is Q where $Q \in \mathbb{C}^{n\times n}$ and $\operatorname{rank}(Q) \le r$, the minimal measurement number m is $4nr - 4r^2$. Since (1.3) is non-convex, it is challenging to solve it [18]. However, under a certain restricted isometry property (RIP), this problem can be relaxed to a nuclear norm minimization problem which is a convex programming and can be solved efficiently [4, 22].

Noting that $M := XX^\top$ is a low rank matrix, we can recast (1.1) as a rank minimization problem. This means that we can use the nuclear norm minimization to recover the matrix M and hence X:

$$\begin{aligned} \min_{Z\in\mathcal{H}_n} \quad & \|Z\|_* \\ \text{subject to} \quad & \operatorname{tr}(A_i Z) = y_i, \quad i = 1,\dots,m, \end{aligned} \tag{1.4}$$

where $\mathcal{H}_n := \{Q \in \mathbb{R}^{n\times n} : Q = Q^\top\}$ and $A_i = a_i a_i^*$. Problem (1.4) was studied in [7, 16] with proving that $m \ge Cnr$ Gaussian measurements are sufficient to recover the unknown matrix $M = XX^\top$ exactly. In [21], Rauhut and Terstiege also consider the case where the measurement vectors $a_i, i = 1,\dots,m$ are from a tight frame.

1.2.2. Phase retrieval

Under the setting of $r = 1$, the (1.1) is reduced to phase retrieval problem. Phase retrieval is to recover an unknown vector from the magnitude of measurements, which means to recover a signal $x \in \mathbb{H}^n$ from measurements

$$y_i = |\langle a_i, x\rangle|^2, \quad i = 1,\dots,m, \tag{1.5}$$

where $a_i \in \mathbb{H}^n$ ($\mathbb{H} = \mathbb{C}$ or $\mathbb{R}$) are sampling vectors. This problem is raised in many imaging applications due to the limitations of optical sensors which can only record intensity information, such as X-ray crystallography [14, 19], astronomy [11], diffraction imaging [13, 24]. It has been proved that $m \ge 4n - 4$ Gaussian measurements are sufficient to recover the unknown vector up to a global phase [8]. In recent years, several different algorithms have been proposed to solve it [1, 2, 9, 10, 20]. In [3], Candès et al. design Wirtinger flow algorithm for phase retrieval which solves the following non-convex optimization problem

$$\min_{u\in\mathbb{C}^n} \frac{1}{4m}\sum_{i=1}^{m}(y_i - |a_i^* u|^2)^2 \tag{1.6}$$

and prove that the algorithm converges to the true signal up to a global phase with high probability provided the measurement vectors are $m = O(n\log n)$ Gaussian measurements.