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One-Step Hybrid Block Method Containing Third Derivatives and Improving Strategies for Solving Bratu's and Troesch's Problems

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Abstract. In this paper, we develop a one-step hybrid block method for solving boundary value problems, which is applied to the classical one-dimensional Bratu's and Troesch's problems. The convergence analysis of the new technique is discussed, and some improving strategies are considered to get better performance of the method. The proposed approach produces discrete approximations at the grid points, obtained after solving an algebraic system of equations. The solution of this system is obtained through a homotopy-type strategy used to provide the starting points needed by Newton's method. Some numerical experiments are presented to show the performance and effectiveness of the proposed approach in comparison with other methods that appeared in the literature.

AMS subject classifications: 65L10, 65L20

Key words: Ordinary differential equations, Bratu's problem, Troesch's problem, boundary value problems, hybrid block method, homotopy-type strategy.

1. Introduction

This paper aims at obtaining numerical solutions for second-order boundary value problems (BVPs) where the differential equation is of the special form

$$y''(x) = f(x, y(x)), \quad x \in [a, b], \tag{1.1}$$

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subject to Dirichlet boundary conditions given by

$$y(a) = y_a, \quad y(b) = b_b, \tag{1.2}$$

although, instead of the above ones, mixed boundary conditions of the form

$$g_1(y(a), y'(a)) = v_a, \quad g_2(y(b), y'(b)) = v_b \tag{1.3}$$

might be considered.

One of those problems is the well-known Bratu's problem, which is given by

$$y''(x) + \lambda \mathrm{e}^{y(x)} = 0, \quad y(0) = y(1) = 0, \quad x \in [0,1]. \tag{1.4}$$

According to Jacobsen and Schmitt [13], and Boyd [2], the theoretical solution for this problem is given by

$$y(x) = -2\log\left[\frac{\cosh\left(\left(x-\frac{1}{2}\right)\frac{\theta}{2}\right)}{\cosh\left(\frac{\theta}{4}\right)}\right], \tag{1.5}$$

where θ is the solution of the algebraic equation

$$\theta = \sqrt{2\lambda}\cosh\left(\frac{\theta}{4}\right).$$

In addition, we will also consider the Troesch's problem, given by

$$y''(x) = \lambda\sinh(\lambda\, y(x)), \quad y(0) = y(1) = 0, \quad x \in [0,1]. \tag{1.6}$$

The close form of the solution for the problem in (1.6) is presented in Khuri [18] as follows

$$y(x) = \frac{2}{\lambda}\sinh^{-1}\left[\frac{y'(0)}{2}\,\mathrm{sc}\left(\lambda x|1-\frac{1}{4}\left(y'(0)\right)^2\right)\right], \tag{1.7}$$

where $y'(0) = 2(1-\mathrm{m})^{1/2}$ and the constant m satisfies the following transcendental equation

$$\frac{\sinh\left(\frac{\lambda}{2}\right)}{(1-\mathrm{m})^{1/2}} = \mathrm{sc}(\lambda|m), \tag{1.8}$$

where $\mathrm{sc}(\lambda|m)$ stands for one of the elliptic Jacobi functions (see [9]).

Problems (1.5) and (1.6) arise in engineering and science. For example, Bratu's problems of the form (1.4) emerge in the thermal reaction process in flammable non-deformable material, like the strong fuel ignition (see [13, 16]). It additionally shows up in the electro-spinning process for the generation of ultra-fine polymer fibers (see [37]), the Chandrasekhar model of the extension of the universe, chemical reactor theory, and nanotechnology (see [26] and the references therein for more details). Troesch's problem in (1.6) is a two-point nonlinear BVP that emerges in the control of a plasma segment by radiation weight and in the theory of gas porous electrodes. This problem was presented and defined for the first time by Troesch in [35].