JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **33**, No. 3, pp. 193-207

doi: 10.4208/jpde.v33.n3.1
September 2020

Well-Posedness of Solutions for Sixth-Order Cahn-Hilliard Equation Arising in Oil-Water-Surfactant Mixtures

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Received 19 March 2020; Accepted 18 April 2020

Abstract. In this paper, by using the L_p-L_q-estimates, regularization property of the linear part of $e^{-t\Delta^3}$ and successive approximations, we consider the existence and uniqueness of global mild solutions to the sixth-order Cahn-Hilliard equation arising in oil-water-surfactant mixtures in suitable spaces, namely $C^0([0,T];\dot{W}^{2,\frac{N(l-1)}{2}}(\Omega))$ when the norm $\|u_0\|_{\dot{W}^{2,\frac{N(l-1)}{2}}(\Omega)}$ is sufficiently small.

AMS Subject Classifications: 35K55; 35A01; 35B45

Chinese Library Classifications: O175.27

Key Words: Global solution; sixth order Cahn-Hilliard equation; uniqueness; oil-water-surfactant mixtures.

1 Introduction

In [1, 2], in order to describe the dynamics of phase transitions in ternary oil-water-surfactant systems, Gompper et al. introduced the free energy functional

$$\mathcal{F}\{u\}=\int_\Omega G(u,\nabla u,\Delta u)\mathrm{d}x, \tag{1.1}$$

with the density given by

$$G(u,\nabla u,\Delta u)=\int_0^u f(s)\mathrm{d}s+\frac{1}{2}a(u)|\nabla u|^2+\frac{\delta}{2}|\Delta u|^2,$$

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where $u(x,t)$ is the scalar order parameter which is proportional to the local difference between oil and water concentrations. The function $a(u)$ is the first gradient energy coefficient which may be of arbitrary sign, δ denotes the mobility and the second gradient energy coefficient and $F(u)\equiv\int_0^u f(s)\mathrm{d}s$ denotes the multiwell volumetric free energy density(see [3–5]). The properties of the amphiphile and its concentration enter model (1.1) implicitly via the form of the functions $F(u)$ and $a(u)$ as well as the magnitude of constant $\delta>0$ (see [3]).

Writing mass conservation, i.e.

$$\frac{\partial u}{\partial t}=-\mathrm{div}j,$$

with the mass flux j given by

$$j=-M\nabla\mu,$$

where M is the mobility and μ is the chemical potential difference between the oil and water phases. Note that the chemical potential is defined by the constitutive equation

$$\mu=\frac{\delta\mathcal{F}\{u\}}{\delta u},$$

where $\frac{\delta\mathcal{F}\{u\}}{\delta u}$ is the first variation of the function $\mathcal{F}\{u\}$, we end up with the following sixth order Cahn-Hilliard type equation:

$$u_t=\mathrm{div}(M\nabla\mu), \tag{1.2}$$

$$\mu=\delta\Delta^2u-a(u)\Delta u-\frac{1}{2}a'(u)|\nabla u|^2+f(u). \tag{1.3}$$

There are some literatures concerned with the initial boundary value problem of equation (1.2)-(1.3). For example, Pawlow and Zajaczkowski [3] proved the existence of unique global smooth solution which depends continuously on the initial datum. In [6], applying the approach based on the Bäcklund transformation and the Leray-Schauder fixed point theorem, Pawlow and Zajaczkowski proved the global unique solvability of the problem in the Sobolev space $H^{6,1}(\Omega\times(0,T))$ under the assumption that the initial datum is in $H^3(\Omega)$ whereas previously $H^6(\Omega)$-regularity was required. Moreover, Liu et al. [7–9] studied time periodic solutions and optimal control problems for the initial boundary value problem of such equation.

Latterly, Schimperna and Pawlow [4] discussed the existence, uniqueness and parabolic regularization of a weak solution to the initial boundary value problem of equation (1.2)-(1.3) together with the viscous term $-\Delta u_t$. In [5], by Leray-Schauder fixed point theorem and suitable estimates, the existence and uniqueness of a global in time regular solution for the sixth order viscous Cahn-Hilliard equation with two viscous terms $-\Delta u_t$ and $\Delta^2 u_t$ are studied.