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DYNAMICAL BEHAVIORS OF ATTRACTION-REPULSION CHEMOTAXIS MODEL

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Abstract. A free boundary problem for the chemotaxis model of parabolic-elliptic type is investigated in the present paper, which can be used to simulate the dynamics of cell density under the influence of the nonlinear diffusion and nonlocal attraction-repulsion forces. In particular, it is shown for supercritical case that if the initial total mass of cell density is small enough or the interaction between repulsion and attraction cancels almost each other, the strong solution for the cell density exists globally in time and converges to the self-similar Barenblatt solution at the algebraic time rate, and for subcritical case that if the initial data is a small perturbation of the steady-state solution and the attraction effect dominates the process, the strong solution for cell density exists globally in time and converges to the steady-state solution at the exponential time rate.

Key words. Chemotaxis, free boundary problem, Barenblatt solution, steady-state solution.

1. Introduction

Chemotaxis is the widespread phenomena in nature, for instance, the directional movement of biological cells, bacteria or organisms in response to chemical signals in the environment, including the positive (chemoattractive) chemotaxis and negative (chemorepulsive) chemotaxis. The first mathematical model is heuristically derived by Patlak [23] and later by Keller and Segel [10, 11] respectively to study the nonlocal aggregation process of cellular slime molds Dictyostelium Discoidium due to chemical cyclic adenosine monophosphate

$$
\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), & \mathbf{x} \in \Omega,\ t \geq 0, \\ \tau v_t = \Delta v + u - av, \end{cases} \tag{1}
$$

where $u = u(\mathbf{x}, t)$ and $v = v(\mathbf{x}, t)$ stand for, the density of cells and concentration of chemoattractant respectively. The non-negative parameter a denotes the mortality rate of chemical, and the parameter τ equals zero or one. Since then, this classical chemotaxis model (1) has been generalized to simulate the biological or medical phenomena [24], such as the bacteria aggregation [29], cancer invasion [2, 33] and so on.

For the sake of simulation the local repulsion of cells in biological or medical phenomena, for example, the volume exclusion or population pressure when cells are packed. Wakita et al [30] observed that the diffusive coefficient is depended on the cell density of bacterial colony through experiment. Kawasaki et al [9] introduced the porous media type bacterial diffusion by modeling spatio temporal patterns of Bacillus subtilis. Therefore, the following Patlak-Keller-Segel model with nonlinear degenerate diffusion could be taken into consideration,

$$
\begin{cases} u_t = \Delta u^m - \nabla \cdot (u \nabla v), & m > 1, \quad \mathbf{x} \in \Omega,\ t \geq 0, \\ \tau v_t = \Delta v + u - av, \end{cases} \tag{2}
$$

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where the diffusive component $m > 1$ denotes the slow diffusion. Topaz et al [27,28] and Carrillo et al [3,4] studied the model (2) to take into accounting over-crowding effects.

In order to model the aggregation of microglia in the central nervous system observed in Alzheimers disease and the quorum-sensing behaviour due to the interaction of chemoattractant and chemorepellent in the chemotaxis process, Luca et al [20] (the diffusive component $m = 1$) and Painter et al [22] (the diffusive component $m > 1$) introduced the following attraction-repulsion chemotaxis model as

$$(3)\qquad \begin{cases} u_t = \Delta u^m - \nabla \cdot (a_1 u \nabla v) + \nabla \cdot (b_1 u \nabla w), & \mathbf{x} \in \Omega,\ t \geq 0, \\ \tau_1 v_t = \Delta v + a_2 u - a_3 v, \\ \tau_2 w_t = \Delta w + b_2 u - b_3 w, \end{cases}$$

where $u = u(\mathbf{x}, t)$, $v = v(\mathbf{x}, t)$, and $w = w(\mathbf{x}, t)$ stand for the density of cells, concentration of chemoattractant and chemorepellent respectively. The diffusive component $m \geq 1$, the non-negative parameters a_i and b_i $(i = 1, 2, 3)$ denote the sensitivity of cells to the chemoattractant and chemorepellent, and the growth and mortality rates of the chemicals respectively, the parameters $\tau_1, \tau_2 = 0, 1$.

The interaction between chemorepellent and chemoattractant is rather complicated which makes it difficult to analyze the mathematical properties of the solution to the system (3). Yet, there are also important progresses made recently on the well-posedness and dynamical behaviors of the solution to the system (3), refer to, for instance, [6–8, 14–18, 18, 19, 21, 25, 26, 31, 32, 34] and the references therein. In particular, the existence of classical solution or the stability of the steady-state solution had been proved in [6–8, 14–16, 18, 19, 26] for the linear diffusion case $m = 1$. In the case that the repulsion effect dominates the process (i.e., $a_1 a_2 - b_1 b_2 < 0$), the global existence of classical solutions was shown to the system (3) in one-dimensional [7, 16, 19] or multi-dimensional bounded domain with the Neumann boundary condition [8, 16, 18, 26], and the long-time convergence of the global classical solutions to the steady-state solution were proved in [7, 15, 18, 19, 26]. Similar results were also established for multi-dimensional Cauchy problem to the system (3), concerned with the global existence of classical solution and long-time convergence to the corresponding steady-state solution [25]. However, in the case that the attraction effect dominates the process (i.e., $a_1 a_2 - b_1 b_2 > 0$), there is a critical initial mass $M = \frac{8\pi}{a_1 a_2 - b_1 b_2}$ as $\tau_1 = \tau_2 = 0$ or $M = \frac{4\pi}{a_1 a_2 - b_1 b_2}$ as $\tau_1 = 1, \tau_2 = 0$ so that the classical solution to the system (3) in two-dimensional bounded domain with the Neumann boundary condition either existed globally in time [6,8] or blew up in finite time [8, 14, 26], depending on whether the initial total mass is larger than M or not. Similar results had also been shown for two-dimensional Cauchy problem to the system (4), related to the blow-up in finite time [25] or global existence of the classical solution [21]. For the nonlinear diffusion case $m > 1$, there are also important results shown in [17,31,32,34] to the system (3) in multi-dimensional bounded domain with the Neumann boundary condition. For instance, the global existence of weak solutions [31,34] or classical solutions [17,32] had been investigated either for $a_1 a_2 - b_1 b_2 < 0$ and $m \neq 2 - \frac{2}{n}$, or for $a_1 a_2 - b_1 b_2 > 0$ and $m > 2 - \frac{2}{n}$. Nevertheless, there existed a class of spherically symmetric weak solutions in three-dimensions which blew up in finite time [17] as it holds $m = 2 - \frac{2}{n}$.

However, although the important achievements have been obtained as above, there are few studies on the global existence and dynamical behaviors of strong solution to the congested motion problem with homogeneous nonlinear degenerate diffusion for fixed component $m > 1$. Li et al [13] have studied the free boundary