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MULTILEVEL FINITE VOLUME METHODS FOR 2D INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

J. K. DJOKO, H. H. GIDEY*, AND B. D. REDDY

Abstract. In this work, implicit and explicit multilevel finite volume methods have been constructed to solve the 2D Navier-Stokes equation with specified initial condition and boundary conditions. The multilevel methods are applied to the pressure-correction projection method using space finite volume discretization. The convective term is approximated by a linear expression that preserves the physical property of the continuous model. The stability analysis of the numerical methods have been discussed thoroughly by making use of the energy method. Numerical experiments exhibited to illustrate some differences between the new (multilevel) and conventional (one-level) schemes.

Key words. Navier-Stokes equations, stability, multilevel finite volume method.

1. Introduction

Let $\Omega = (0, L_1) \times (0, L_2) \subseteq \mathbb{R}^2$ be an open and bounded region in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$ and points denoted by $(x, y) \in \overline{\Omega} = \Omega \cup \partial\Omega$. Let $\langle \cdot, \cdot \rangle$ denote the $L^2(\Omega)$ inner product of vectors or matrix fields on Ω, depending on the context; i.e.,

$$\langle \mathbf{u}, \mathbf{u} \rangle = \int_\Omega \mathbf{u} \cdot \mathbf{v} \, d\Omega, \tag{1}$$

where $\mathbf{u}$ and $\mathbf{v}$ are arbitrary vectors on Ω. The associated L^2-norm is denoted by $\| \cdot \| = \sqrt{\langle \cdot, \cdot \rangle}$. The spatial velocity field of the fluid filling the region $\overline{\Omega}$ is denoted by $\mathbf{u}(x, y, t)$, where $t \in [0, T]$, $T \in \mathbb{R}_+$.

The Navier-Stokes equations governing the dynamics of the viscous incompressible and homogeneous fluids is written in the generic form [1]

$$\mathbf{u}_t + B(\mathbf{u})\mathbf{u} = -\nabla p + \nu \Delta \mathbf{u} + f, \qquad \text{in } \Omega \tag{2}$$

$$\operatorname{div} \mathbf{u} = 0, \tag{3}$$

associated with the following boundary conditions and initial data:

$$\mathbf{u} = 0, \text{ on } \partial\Omega \tag{4}$$

$$\mathbf{u}|_{t=0} = \mathbf{u}_0 \quad \text{in} \quad \Omega, \tag{5}$$

where $B(\mathbf{u})\mathbf{u}$ is the convective term, $\nu > 0$ is the kinematic shear viscosity, p is a pressure field arising from incompressibility constraint div $\mathbf{u} = 0$ and f is applied body force.

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*Corresponding author.

Let $\mathcal{H}$ be the space of $L^2(\Omega)$-smooth vector fields tangent to the boundary $\partial\Omega$ and denote by $\mathcal{H}_{\text{div}}$ the subspace of divergence-free vector fields:

$$\mathcal{H} := \big\{\mathbf{u} \in L^2(\Omega)^2 : \mathbf{u}(0,y,t) = \mathbf{u}(L_1,y,t) = \mathbf{u}(x,0,t) = \mathbf{u}(x,L_2,t) = 0, \tag{6}$$

$$x \in [0,L_1], y \in [0,L_2]\} \tag{7}$$

$$\mathcal{H}_{\text{div}} := \{\mathbf{u} \in \mathcal{H} : \text{ div } \mathbf{u} = 0\}. \tag{8}$$

In this study, we consider the standard form of the convective term, i.e, $B(\mathbf{u})\boldsymbol{\eta} = (\mathbf{u}\cdot\nabla)\boldsymbol{\eta}$, for any smooth $H^1(\Omega)$-vector field $\boldsymbol{\eta}$, with associated pressure field denoted by p. Using integration by parts, we obtain

$$\langle B(\mathbf{u})\boldsymbol{\eta}_1, \boldsymbol{\eta}_2\rangle = -\langle \boldsymbol{\eta}_1, B(\mathbf{u})\boldsymbol{\eta}_2\rangle - \langle \text{div}\mathbf{u}\,\boldsymbol{\eta}_1, \boldsymbol{\eta}_2\rangle + \int_\Gamma (\mathbf{u}\boldsymbol{\eta}_1\cdot\boldsymbol{\eta}_2)(\mathbf{u}\cdot\mathbf{n})d\Gamma, \tag{9}$$

for arbitrary $H^1(\Omega)$-smooth vector fields $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2$ on Ω.

$$\langle B(\mathbf{u})\boldsymbol{\eta}_1, \boldsymbol{\eta}_2\rangle = -\langle \boldsymbol{\eta}_1, B(\mathbf{u})\boldsymbol{\eta}_2\rangle, \mathbf{u} \in \mathcal{H}_{\text{div}}. \tag{10}$$

$$\langle B(\mathbf{u})\boldsymbol{\eta}, \boldsymbol{\eta}\rangle = 0, \text{ for any } H^1(\Omega) \text{ smooth vector field } \boldsymbol{\eta}, \tag{11}$$

only holds if the velocity field is divergence-free; $\mathbf{u} \in \mathcal{H}_{\text{div}}$.

Integrating equation (3) over a control volume and converting the volume integral to a surface integral gives

$$\int_\Omega \text{div } \mathbf{u}\,dx\,dy = \oint_S \mathbf{u}\cdot\mathbf{n}\,dx\,dy = 0. \tag{12}$$

This shows that the inflow must be equal to the outflow.

Our objective is to construct multilevel finite volume methods based on the work in [2-4] to compute the numerical solution of (2)-(5). Multilevel methods were introduced to improve calculation speed in the simulation of complex physical phenomena while maintaining good accuracy [3-8]. We construct implicit and explicit finite volume methods based on the work of Appadu et al. [2] and Bousquet et al. [4]. The schemes we construct are easy to implement and the convective term $B(\mathbf{u})\mathbf{u}$ is approximated such that the discrete analogue of the property (11) holds.

Our work can also be seen as continuation of investigations started in [1] because in a way we are concerned with the stability of the new schemes that should preserve (11). The main difference with the former investigation is that we are dealing here with multilevel scheme, hence stability analysis is more complex, even with the use of a simpler technique (energy method). We do not discuss existence of solutions of the schemes formulated because we are dealing with linear scheme (for the implicit multilevel method) and explicit multilevel method. Hence solvability of the implicit scheme is a consequence of Lax-Milgram's result in the discrete setting.

The next section is devoted to space discretization and some properties that are helpful to our study. In section 3, we are concerned with the multilevel discretization and time stepping algorithm. In sections 4 and 5, we present the implicit and explicit multilevel finite volume methods, respectively and analyse their stability. In section 6, we present the numerical results obtained from the two multilevel methods and these results are compared with the full one-level finite volume methods on the fine mesh and coarse mesh. Concluding remarks and some open questions are reported in section 7.