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THE TIME HIGH-ORDER ENERGY-PRESERVING SCHEMES FOR THE NONLOCAL BENJAMIN-ONO EQUATION

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Abstract. The new time high-order energy-preserving schemes are proposed for the nonlocal Benjamin-Ono equation. We get the Hamiltonian system to the nonlocal model, and it is then discretized by a Fourier pseudospectral method in space and the Hamiltonian boundary value method (HBVM) in time. This approach has high order of convergence in time and conserves the total mass and energy in discrete forms. We further develop a time second-order energy-preserving scheme and a time fourth-order energy-preserving scheme for the nonlocal Benjamin-Ono equation. Numerical experiments test the proposed schemes with a single solitary wave and the interaction of two solitary waves. Results confirm the accuracy and conservation properties of the schemes.

Key words. Nonlocal Benjamin-Ono equation, Hamiltonian boundary value method (HBVM), time high-order, energy preserving, Fourier pseudospectral method.

1. Introduction

Recently, there are more increasing interests in studying the problems of nonlocal partial differential equations in physics, mechanics, biology, materials science, and imaging science, etc. We consider the nonlocal Benjamin-Ono equation, which is a nonlocal partial differential equation arising in the study of long internal gravitation waves in deep stratified fluids and modelling the propagation of nonlinear dispersive waves ([3, 12, 14]).

The nonlocal Benjamin-Ono equation describes the remarkable properties of nonlinear dispersive wave propagation, that they permit stable, localized waveform solutions travelling at constant speeds, called solitary waves [11]. When two solitary waves overtake each other, they emerge from the interaction without any changes in shape and speed. James and Weideman [13] proposed a pseudospectral method for the Benjamin-Ono equation by the Hilbert transform, which is a convolution, reduces to a product under the spectral discretization. Boyd and Xu [9] compared three pseudospectral methods based on the Fourier, radial basis and rational orthogonal basis functions for the Benjamin-Ono equation and obtained exponential convergence in space. Thomee and Murthy [15] solved the Benjamin-Ono equation by a finite difference approximations in space and the Crank-Nicolson approximation in time. This approach has the accuracy order $O(h^2 + \Delta t^2)$. Although the spectral methods are commonly used to solve the Benjamin-Ono equation, they do not conserve the physical invariants if the system is integrated in time by non-conservative integrators such as the standard Runge-Kutta methods or multi-step methods. As a result, dissipative errors will be introduced and the shapes and speeds of solitary solutions will change in numerical simulations. Therefore, it is very important and dificult to develop time high-order energy-preserving numerical schemes to the nonlocal Benjamin-Ono equation.

Brugnano and Iavernaro et al [6, 7, 8] proposed a class of structure-conserved method, namely the Hamiltonian boundary value methods (HBVMs) that yield

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the conservation for Hamiltonian invariants represented by polynomial functional of arbitrarily high-degree. Moreover, the methods are shown to be symmetric, precisely A-stable, and can have arbitrarily high-order accuracy. The methods have been extensively applied to simulate Hamiltonian partial differential equations, such as the semilinear wave equation [4], the nonlinear Schrödinger equation [2], the KdV equation [5] and the modified KdV equation [16]. However, to our best knowledge, the HBVMs have not been applied to approximate any *nonlocal* dispersive partial differential equations, such as the nonlocal Benjamin-Ono equation.

In this paper, we develop time high-order energy-preserving schemes for the nonlocal Benjamin-Ono equation. We first get the Hamiltonian system to the nonlocal model of Benjamin-Ono equation. We then discretize the nonlocal Benjamin-Ono equation in space by the Fourier pseudospectral method. We show that the resulting semi-discrete system can be written as a Hamiltonian system. We integrate the corresponding discrete Hamiltonian system with the HBVM approach to obtain a time second-order scheme and a time fourth-order scheme, both preserving the mass and energy indiscrete forms. Numerical experiments are given to show the preserving properties and convergence orders of the schemes and to show the physical phenomenon of the interaction of solitary waves of the nonlocal models.

This paper is organized as follows. In Section 2, we present the nonlocal model of Benjamin-Ono equation and derive out its Hamiltonian system. In Section 3, we derive the Runge-Kutta formulation of the HBVMs. In Section 4, we introduce the basic properties of the Fourier pseudospectral method and obtain the time second-order and time fourth-order energy-preserving schemes. We show numerical experiments in Section 5 and some conclusions are addressed in Section 6.

2. Nonlocal model of Benjamin-Ono equation and its Hamiltonian system

Consider the nonlocal model of the Benjamin-Ono equation [15]

$$\begin{cases} u_t + uu_x - Hu_{xx} = 0, \quad x \in [-L, L], t \in [0, +\infty), \\ u(x,0) = u_0(x), \quad x \in [-L, L], \end{cases} \tag{1}$$

with $u(x+2L,t) = u(x,t)$ and H is the Hilbert transform defined by

$$\begin{aligned} Hu(x) &= \frac{1}{\pi} P.V. \int_{-\infty}^{+\infty} \frac{u(x-y)}{y} dy \\ &= \frac{1}{2L} P.V. \int_{-L}^{L} \cot(\frac{\pi}{2L} y) u(x-y) dy \end{aligned} \tag{2}$$

for the periodic function $u(x)$. For details of the periodic Hilbert transform and theoretical analysis of (1), we refer to [15] and the references therein. It can be shown that the periodic problem (1) has many invariants, such as

$$\begin{aligned} \mathcal{M} &= \int_{-L}^{L} u dx, \\ \mathcal{I} &= \frac{1}{2} \int_{-L}^{L} u^2 dx, \\ \mathcal{H} &= \frac{1}{6} \int_{-L}^{L} \left[u^3 + 3u_x H(u) \right] dx. \end{aligned} \tag{3}$$

These invariants are usually referred as mass, momentum and energy, respectively.