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MODIFYING THE SPLIT-STEP θ-METHOD WITH HARMONIC-MEAN TERM FOR STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we design a class of general split-step methods for solving Itô stochastic differential systems, in which the drift or deterministic increment function can be taken from special ordinary differential equations solver, based on the harmonic-mean. This method is justified to have a strong convergence order of $\frac{1}{2}$. Further, we investigate mean-square stability of the proposed method for linear scalar stochastic differential equation. Finally, some examples are included to demonstrate the validity and efficiency of the introduced scheme.

Key words. Itô stochastic differential system, split-step method, ODE solver, harmonic-mean, strong convergence, mean-square stability.

1. Introduction

Many phenomena in various branches of science like physics, chemistry and engineering can be modeled more efficiently by the stochastic differential equations (SDEs) [3,5,6,15]. Since analytical solutions of SDEs are generally not available, we are forced to use numerical methods that give approximated solutions [8,9,13,15,21, 25,29,40]. First attempt in this direction was made by Maruyama [17], who established the well-known Euler-Maruyama (EM) method, then Milstein [18] presented an important numerical scheme with faster convergence than EM method [9,10,35]. Based on EM and Milstein methods, many numerical schemes have been presented and developed later, see for example [2, 11, 12, 19, 22, 23, 30, 36, 37].

In [24], Platen and Wagner proposed a stochastic generalization of the Taylor formula for Itô diffusions. This generalization, called the Itô-Taylor expansions, was based upon the use of multiple stochastic integrals. The Itô-Taylor expansions are characterized by the choice of multiple integrals which appear in them. Many numerical methods based on Itô-Taylor expansions have been presented for simulating the approximate solutions to SDEs [15,20]. In this paper we will consider numerical methods for strong solution of Itô stochastic differential systems of the form

$$\mathrm{d}X(t) = f(X(t))\mathrm{d}t + \sum_{j=1}^{m} g_j(X(t))\mathrm{d}B^j(t), \quad X(t_0) = X_0, \ t \in [t_0, T], \tag{1}$$

where $X \in \mathbb{R}^d$, $f : \mathbb{R}^d \to \mathbb{R}^d$, is a drift vector, $g = (g_1, \ldots, g_m) : \mathbb{R}^d \to \mathbb{R}^{d\times m}$ is a diffusion matrix and $B = (B_1, \ldots, B_m)^T$ is an m–dimensional Brownian motion process. Similarly to contributions [8, 25, 34], we design and analyze the strong convergence of a class of general split-step methods for solving the Itô stochastic differential system (1).

Nowadays, stability is judged better to account the efficiency of numerical methods for solving SDEs. Several kinds of these stabilities have been proposed in [13,14,16].

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Throughout this paper the so-called mean-square (MS) stability will be considered. This kind of stability, which is based on the second statistical moment of the (exact or numerical) solution, has been considered in the literature so far [4, 7, 26, 33, 39]. In order to discuss mean-square stability properties of our proposed method, we will focus on the special linear scalar Itô test equation

$$\mathrm{d}X(t) = aX(t)\mathrm{d}t + bX(t)\mathrm{d}B(t), \quad t \geq t_0, \quad X(t_0) = X_0, \tag{2}$$

where a, $b \in \mathbb{C}$ and $X_0 \neq 0$ are constants. For numerical step size h and $a, b \in \mathbb{R}$, Saito and Mitsui [26] plot MS-stability region in the $(\overline{h}, k)$-plane, with $\overline{h} = ah$, $k = -\frac{b^2}{a}$. Then, Higham [11] performed the analysis in the (x, y)-plane with $x = ah$, $y = b^2 h$, which has been accepted by researchers for presenting MS-stability domain of numerical stochastic methods [4, 7, 11, 33]. Moreover, some MS-stability domains have been plotted with $x = ah$, $y = b\sqrt{h}$ and $a, b \in \mathbb{R}$ (see [8, 30, 34, 37, 39]).

The paper is organized as follows. Section 2 is devoted to introducing the proposed method. Convergence properties of the method are discussed in Section 3. Mean-square stability properties and numerical results of the method are reported in Sections 4 and 5, respectively.

2. General split-step method

For solving stochastic differential system (1), thereupon we present general split-step methods, based on EM numerical scheme, of the form

$$\begin{cases} \overline{Y}_k & = Y_k + h\varphi(Y_k, \overline{Y}_k), \\ Y_{k+1} & = \overline{Y}_k + \sum_{j=1}^{m} g_j(\overline{Y}_k)\Delta B_k^j, \end{cases} \tag{3}$$

where $\varphi(Y_k, \overline{Y}_k)$ is an increment function of the deterministic ordinary differential equation (ODE) solver. This idea was first presented by Higham in [12], as a modification of the classical EM method, which is usually referred to as split-step methods. This approach is a class of fully implicit methods which allows us the incorporation of implicitness in the stochastic part of the system with relatively little additional cost. Then, Wang and Li in [36] presented two types of split-step methods, drifting split-step Euler and diffused split-step Euler methods, for SDEs by a single noise term. Ding et al. [4] have analysed the split-step θ-methods for solving nonlinear non-autonomous SDEs. Guo et al. in [7] improved split-step θ-methods for solving SDEs systems by a single noise term. Recently, error corrected EM method, which is constructed by adding an error correction term to the EM method, was introduced in [39].

Instead of using the above methods on the increment function, we replace them by a method based on different means to solve ODEs. In this paper, based on the concept of averaging the harmonic-mean functional [27, 38], we consider the ODE solver in the form,

$$\varphi(Y_k, \overline{Y}_k) = (1-\theta)f(\overline{Y}_k) + 2\theta\left(f^{-1}(Y_k) + f^{-1}(\overline{Y}_k)\right)^{-1}, \quad \theta \in [0, 1]. \tag{4}$$

Here $f^{-1}(\cdot) = \frac{1}{f(\cdot)}$ and we assume that $f(Y_k) + f(\overline{Y}_k) \neq 0$. The choice $\theta = 0$ and $m = 1$ becomes the method introduced in [12]. Note that by inserting ODEs solver harmonic-mean θ (HMT) (4) into general split-step method (3), we have the