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Boundary Values of Generalized Harmonic Functions Associated with the Rank-One Dunkl Operator

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Dedicated to Professor Weiyi Su on the occasion of her 80th birthday

Abstract. We consider the local boundary values of generalized harmonic functions associated with the rank-one Dunkl operator D in the upper half-plane $\mathbb{R}^2_+ = \mathbb{R} \times (0, \infty)$, where

$$(Df)(x) = f'(x) + (\lambda/x)[f(x) - f(-x)]$$

for given $\lambda \geq 0$. A C^2 function u in $\mathbb{R}^2_+$ is said to be λ-harmonic if $(D_x^2 + \partial_y^2)u = 0$. For a λ-harmonic function u in $\mathbb{R}^2_+$ and for a subset E of $\partial\mathbb{R}^2_+ = \mathbb{R}$ symmetric about y-axis, we prove that the following three assertions are equivalent: (i) u has a finite non-tangential limit at $(x, 0)$ for a.e. $x \in E$; (ii) u is non-tangentially bounded for a.e. $x \in E$; (iii) $(Su)(x) < \infty$ for a.e. $x \in E$, where S is a Lusin-type area integral associated with the Dunkl operator D.

Key Words: Dunkl operator, Dunkl transform, harmonic function, non-tangential limit, area integral.

AMS Subject Classifications: 42B20, 42B25, 42A38, 35G10

1 Introduction and main results

For given $\lambda > 0$, the rank-one Dunkl operator on the line $\mathbb{R}$ is defined by

$$(Df)(x) = f'(x) + \frac{\lambda}{x}(f(x) - f(-x)).$$

A C^2 function u in the upper half-plane $\mathbb{R}^2_+ = \mathbb{R} \times (0, \infty)$ is said to be λ-harmonic if $\Delta_\lambda u = 0$, where

$$\Delta_\lambda = D_x^2 + \partial_y^2.$$

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The operator Δ_λ is called the λ-Laplacian, and can be written explicitly by

$$(\Delta_\lambda u)(x,y) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{2\lambda}{x}\frac{\partial u}{\partial x} - \frac{\lambda}{x^2}\left(u(x,y) - u(-x,y)\right).$$

Some aspects of harmonic analysis in the upper half-plane $\mathbb{R}^2_+$ associated to the Dunkl operator D were studied in [25] and their analogues in the unit disk $\mathbb{D}$, associated with Dunkl-Gegenbauer expansions, were developed in [26]. These are generalizations of the seminal work of Muckenhoupt and Stein [31] on the Bessel operator and the Gegenbauer expansions. In this paper we study the local existence of boundary values of λ-harmonic functions in the upper half-plane $\mathbb{R}^2_+$.

It is well known that, if u is a harmonic function in the unit disk $\mathbb{D}$ and E is a subset of positive measure of the boundary $\partial\mathbb{D}$, then the existence of non-tangential limit at almost every $e^{i\theta} \in E$ of u can be characterized by non-tangential boundedness of u at almost every $e^{i\theta} \in E$, and also by finiteness of Lusin's area integral of u at almost every $e^{i\theta} \in E$. The former, as a local version of Fatou's theorem, was owed to Privalov [38], and the latter was proved by Marcinkiewicz and Zygmund [30] and Spencer [42]. One of the basic tools in these works is the conformal mapping, which introduces technical difficulties in extending them to more variables and other settings. Calderón [5,6] made a breakthrough and generalized Privalov's theorem and Marcinkiewicz and Zygmund's theorem to Euclidean half-spaces of several variables by the real-variable method. A generalization of the theorem of Spencer [42] to several variables was obtained in Stein [43]. Since then, criteria on existence of non-tangential boundary limits of harmonic functions in many different contexts, in terms of non-tangential boundedness or one-side non-tangential boundedness or finiteness of area integrals have been intensively studied; see, for example, [1-4,7,14-22,24,32-37,39] and [46].

As usual, we denote by $\Gamma_\alpha(x)$ the positive cone of aperture $\alpha > 0$ with vertex $(x,0) \in \partial\mathbb{R}^2_+ = \mathbb{R}$, and $\Gamma^h_\alpha(x)$ the truncated one with height $h > 0$, that is,

$$\Gamma^h_\alpha(x_0) = \{(x,y) \in \mathbb{R}^2_+ : |x - x_0| < \alpha y,\ 0 < y < h\}.$$

For a function u defined in $\mathbb{R}^2_+$ and for $\alpha > 0$, the non-tangential maximal function $u^*_\nabla(x)$ is defined by

$$u^*_\nabla(x) = \sup_{(t,y)\in\Gamma_\alpha(x)} |u(t,y)|;$$

that u has a non-tangential limit at $(x,0)$ means that for every $\alpha > 0$, $\lim u(t,y)$ exists as $(t,y) \in \Gamma_\alpha(x)$ approaching to $(x,0)$; and that u is said to be non-tangentially bounded at $(x,0)$ if $u(t,y)$ is bounded in $\Gamma^h_\alpha(x)$ for some $\alpha, h > 0$. For a C^2 function u in $\mathbb{R}^2_+$, we define the Lusin-type area integral $Su = S_{\alpha,h}u$ for some $\alpha, h > 0$ by

$$(S_{\alpha,h}u)(x) = \left(\int_{\Gamma^h_\alpha(0)} \tau_x(\Delta_\lambda u^2)(-t,y)y^{-2\lambda}|t|^{2\lambda}dtdy\right)^{1/2},$$