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How to Define Dissipation-Preserving Energy for Time-Fractional Phase-Field Equations

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Abstract. There exists a well defined energy for classical phase-field equations under which the dissipation law is satisfied, i.e., the energy is non-increasing with respect to time. However, it is not clear how to extend the energy definition to time-fractional phase-field equations so that the corresponding dissipation law is still satisfied. In this work, we will try to settle this problem for phase-field equations with Caputo time-fractional derivative, by defining a nonlocal energy as an averaging of the classical energy with a time-dependent weight function. As the governing equation exhibits both nonlocal and nonlinear behavior, the dissipation analysis is challenging. To deal with this, we propose a new theorem on judging the positive definiteness of a symmetric function, that is derived from a special Cholesky decomposition. Then, the nonlocal energy is proved to be dissipative under a simple restriction of the weight function. Within the same framework, the time fractional derivative of classical energy for time-fractional phase-field models can be proved to be always nonpositive.

AMS subject classifications: 65M06, 65M12, 74A50

Key words: Phase-field equation, energy dissipation, Caputo fractional derivative, Allen-Cahn equations, Cahn-Hilliard equations, positive definite kernel.

1 Introduction

A fractional time derivative arises when the characteristic waiting time diverges, which models situations involving memory, see, e.g., [1,2]. In recent years, to model memory

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effects and subdiffusive regimes in applications such as transport theory, viscoelasticity, rheology and non-Markovian stochastic processes, there has been an increasing interest in the study of time-fractional differential equations, i.e., differential equations where the standard time derivative is replaced by a fractional one, typically a Caputo or a Riemann-Liouville derivative.

For the models involved Caputo fractional derivative, Allen, Caffarelli and Vasseur [3] studied the regularity of a time-fractional parabolic problem. Their main result is a De Giorgi-Nash-Moser Hölder regularity theorem for solutions in a divergence form equation. In a more recent work [4], they performed regularity study for porous medium flow with both a fractional potential pressure and fractional time derivative. In [5], Luchko and Yamamot discussed the maximum principle for a class of time-fractional diffusion equation with the Caputo time-derivative. In [6], Li, Liu and Wang investigated Cauchy problems for nonlinear time-fractional Keller-Segel equation with the Caputo time-derivative. Some important properties of the solutions including the nonnegativity preservation, mass conservation and blowup behaviors are established.

On the other hand, for the models involved Riemann-Liouville fractional time derivative, Zach [7] investigated the regularity of weak solutions to a class of time fractional diffusion equations and obtained a De Giorgi-Nash type theorem which gives an interior Hölder estimate for bounded weak solutions. In [8], Vergara and Zacher investigated optimal decay estimates by using energy methods; and in [9], they studied instability and blowup properties for Riemann-Liouville time-fractional subdiffusion equations. In [10], Le, McLean and Stynes studied the well-posedness of the solution of the time-fractional Fokker-Planck equation with general forcing.

Most of the works mentioned above are of semi-linearity in space. It is noticed that there exists active research on time-fractional problems with spatial nonlinearity, which arises in practical applications. For example, Allen, Caffarelli and Vasseur [4] considered a time-space fractional porous medium equation with Caputo fractional time derivatives and nonlocal diffusion effects. In [11], Giga and Namba investigated the well-posedness of Hamilton-Jacobi equations with a Caputo fractional time derivative, with a main purpose of finding a proper notion of viscosity solutions so that the underlying Hamilton-Jacobi equation is well-posed. A further study along this line is recently provided by Camilli and Goffi [12]. Their study relies on a combination of a gradient bound for the time-fractional Hamilton-Jacobi equation obtained via nonlinear adjoint method and sharp estimates in Sobolev and Hölder spaces for the corresponding linear problem.

The Cahn-Hilliard model [13] may be the most popular phase-field model whose governing equation is of the form

$$\partial_t\phi-\gamma\Delta\left(-\varepsilon^2\Delta\phi+F'(\phi)\right)=0,\quad x\in\Omega\subset\mathbb{R}^d,\ \ 0<t\le T, \tag{1.1}$$

where ε is an interface width parameter, γ is the mobility, and F is a double-well potential that is usually taken the form $F(\phi)=\frac{1}{4}(1-\phi^2)^2$. The corresponding free energy functional