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On the Construction and Analysis of Finite Volume Element Schemes with Optimal L^2 Convergence Rate

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Abstract. We provide a general construction method for a finite volume element (FVE) scheme with the optimal L^2 convergence rate. The k-(k-1)-order orthogonal condition (generalized) is proved to be a sufficient and necessary condition for a k-order FVE scheme to have the optimal L^2 convergence rate in 1D, in which the independent dual parameters constitute a (k-1)-dimension surface in the reasonable domain in k-dimension.

In the analysis, the dual strategies in different primary elements are not necessarily to be the same, and they are allowed to be asymmetric in each primary element, which open up more possibilities of the FVE schemes to be applied to some complex problems, such as the convection-dominated problems. It worth mentioning that, the construction can be extended to the quadrilateral meshes in 2D. The stability and H^1 estimate are proved for completeness. All the above results are demonstrated by numerical experiments.

AMS subject classifications: 65N12, 65N08, 65N30

Key words: Finite volume, L^2 estimate, sufficient and necessary condition, orthogonal condition.

1. Introduction

The finite volume element method (FVEM) [1–3, 6, 9–11, 14–16, 18, 19, 21, 26, 29–31, 34, 36] is a type of finite volume method (FVM), which is famous for the local conservation property and has been successfully applied to a broad range of problems. Till now, a lot of progress has been made in understanding the stability and H^1 estimate [8, 9, 20, 22, 28, 35, 37], L^2 estimate [7, 13, 17, 23, 24, 32], and superconvergence [4, 5, 25, 33] for the FVEM. Most of the existing results talk about the FVE schemes with symmetric dual meshes.

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This paper focuses on the construction and L^2 analysis for the FVE schemes whose dual meshes could be asymmetric in 1 dimension. Due to the test space of the FVEM is different from the trial space, given a Lagrange k-order trial space, there are different FVE schemes corresponding to the different choices of dual meshes and the piecewise constant test space over it. Nevertheless, not all of these FVE schemes have optimal L^2 convergence rate, including the even-order FVE schemes with uniform dual meshes. Therefore, how to choose the dual meshes is a main issue of the FVE schemes' construction [8, 23, 25, 32, 33, 35, 37]. To the authors' knowledge, current results about the L^2 estimate are mainly focused on sufficient conditions for FVE schemes to have the optimal L^2 convergence rate. At the same time, little progress has been made in understanding the necessary conditions.

In this paper, the k-(k-1)-order orthogonal condition (generalized) is proved to be a sufficient and necessary condition for the FVE schemes to hold the optimal L^2 convergence rate. The k-(k-1)-order orthogonal condition means some orthogonality to the (k-1)-order polynomial space on each element K in the sense of the inner product. It was first proposed in [32] for the FVE schemes with symmetric dual meshes over triangular meshes, and a simple comparative study was made on the number of independent restrictions and the number of independent dual parameters to determine the dual meshes. In the present paper, it is proved that, when the k-(k-1)-order orthogonal condition is satisfied, the k dual parameters form a (k-1)-dimension surface in the reasonable domain in k-dimension. That means, for the FVE schemes holding optimal L^2 convergence rate, their k dual parameters form a (k-1)-dimension surface in k-dimension. In particular, for FVE schemes with symmetric dual meshes, all odd-order schemes have optimal L^2 convergence rate, while not all even-order schemes hold optimal L^2 convergence rate even if they have uniform dual meshes, which are consistent with what we knew before from numerical experiments.

It's worth mentioning that, in the analysis, the dual strategies in different element could be different, which allows more applications of the FVEM. Numerical experiments are presented for some convection-dominated problems (Examples 5.4-5.5). The performances of the standard quadratic finite element (FE) scheme (FE2), the quadratic FVE scheme with Gauss-Lobatto dual strategy (FV2-5), and the quadratic FVE schemes (FV2-6 in Example 5.4 and FV2-7 in Example 5.5) with asymmetric dual meshes are compared over uniform primary meshes. It's shown that the quadratic FVE schemes (FV2-6 and FV2-7) with proper dual strategies perform better than FE2 and FV2-5 in these two examples, which demonstrates the capacity of the FVEM to solve some convection-dominated problems.

Following Section 2 shows the definition of the FVE schemes for the two-points boundary value problem. Section 3 introduces the orthogonal condition and the corresponding equivalent equations, which helps to construct FVE schemes. In Section 4, we present the main result of this paper, a necessary and sufficient condition for a FVE scheme to have the optimal L^2 convergence rate. Numerical experiments, including one on the rectangular meshes, are shown in Section 5 to illustrate our theoretical results. Then, we made conclusion in Section 6 and analyse the stability in Appendix A.