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AN EXPLICIT FORMULA FOR CORNER SINGULARITY EXPANSION OF THE SOLUTIONS TO THE STOKES EQUATIONS IN A POLYGON

BUYANG LI

Abstract. We present corner singularity expansion for the solutions of the Stokes equations in an arbitrary polygon under the stress boundary condition, with explicit expression of its singular part and quantitative estimate of its regular part. In particular, there is a countably infinite set of angles such that a corner with one of these angles would give the solution precisely one additional logarithmic singularity, whose explicit expression is found.

Key words. Corner singularity, regularity, Stokes equation, stress boundary condition, fractional Sobolev space.

1. Introduction

Consider the Stokes equations with the stress boundary condition in a polygon $\Omega \subset \mathbb{R}^2$, i.e.

$$\begin{cases} -\nabla\cdot(2\mathbb{D}(\mathbf{u})-p\mathbb{I})=\mathbf{f} & \text{in } \Omega, \\ \nabla\cdot\mathbf{u}=g & \text{in } \Omega, \\ (2\mathbb{D}(\mathbf{u})-p\mathbb{I})\mathbf{n}=\mathbf{h} & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\mathbb{D}(\mathbf{u})=\frac{1}{2}(\nabla\mathbf{u}+(\nabla\mathbf{u})^T)$ is the stress tensor, and $\mathbf{n}$ is the unit outward normal vector on the boundary $\partial\Omega$. For any given $(\mathbf{f},g,\mathbf{h})\in\mathbf{H}^1(\Omega)'\times L^2(\Omega)\times\mathbf{H}^{1/2}(\partial\Omega)'$, the weak formulation of the problem (1) is to find $(\mathbf{u},p)\in\mathbf{H}^1(\Omega)\times L^2(\Omega)$ such that

$$\begin{cases} 2(\mathbb{D}(\mathbf{u}),\mathbb{D}(\mathbf{v}))_\Omega-(p,\nabla\cdot\mathbf{v})_\Omega=(\mathbf{f},\mathbf{v})_\Omega+(\mathbf{h},\mathbf{v})_{\partial\Omega}, & \forall\ \mathbf{v}\in\mathbf{H}^1(\Omega), \\ (\nabla\cdot\mathbf{u},\psi)_\Omega=(g,\psi)_\Omega, & \forall\ \psi\in L^2(\Omega), \end{cases} \tag{2}$$

under the compatibility conditions $\int_\Omega\mathbf{f}(x)\mathrm{d}x+\int_{\partial\Omega}\mathbf{h}(x)\mathrm{d}\tau=0$. Existence and uniqueness of the weak solution can be proved by using the Lax–Milgram lemma (see [16] or appendix A).

For many purposes, higher regularity of the solution is often needed [3, 8, 11]. However, due to the existence of corners of the domain Ω, for a given right-hand side $(\mathbf{f},g,\mathbf{h})\in\mathbf{H}^{s-1}(\Omega)\times H^s(\Omega)\times\mathbf{H}^{s-1/2}(\partial\Omega)$ the solution $(\mathbf{u},p)$ may not be in $\mathbf{H}^{s+1}(\Omega)\times H^s(\Omega)$ for general $s>0$. It is natural to ask how smooth the solution can be and what specific singularities the solution possesses.

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Under the Dirichlet boundary conditon $\mathbf{u} = 0$ on $\partial\Omega$, Kellogg and Osborn [13] proved that the solution is in $\mathbf{H}^2(\Omega) \times H^1(\Omega)$ when the polygon is convex and $(\mathbf{f}, g) \in \mathbf{L}^2(\Omega) \times H^1(\Omega)$, with an additional condition $g/r \in L^2(\Omega)$, where r denotes the distance from the corners of the domain. Under the compatibility condition

$$g = 0 \text{ at the corners of the polygon if } s > 1, \tag{3}$$

Dauge [7] proved the $\mathbf{H}^{s+1}(\Omega) \times H^s(\Omega)$ regularity for fractional index $s < \operatorname{Re} \lambda_1(\omega)$ in general polygons, where $\lambda_1(\omega)$ is the first non-integer root of the equation

$$\lambda^2 \sin(\omega)^2 - \sin(\lambda\omega)^2 = 0, \tag{4}$$

and ω is the maximal interior angle of the polygon Ω. In general, $\lambda_1(\omega) > 1$ if Ω is a convex polygon, and $\lambda_1(\omega) > 1/2$ if Ω is a non-convex polygon. With more general boundary conditions, Orlt and Sändig [17] found the range of the index s for which the $\mathbf{H}^{s+1}(\Omega) \times H^s(\Omega)$ regularity holds, Guo and Schwab [10] investigated the analytic regularity of the solution in the framework of L^2-based weighted Sobolev spaces, Maz'ya and Rossmann [15] presented regularity estimates of the solution in terms of L^p-based weighted Sobolev and Hölder spaces.

The basic ideas of [7, 10, 13, 17] are restricting the Stokes equations to a cone centered at a corner of the polygon and transforming the Stokes equation to an ODE system (with a parameter $\lambda \in \mathbb{C}$)

$$\mathcal{L}(\lambda)(\widehat{\mathbf{U}}(\lambda), \widehat{q}(\lambda)) = (\widehat{\mathbf{F}}(\lambda), \widehat{G}(\lambda), \widehat{\mathbf{H}}(\lambda))$$

by using the Mellin transform (or Fourier transform), and then studying the spectrum of the operator $\mathcal{L}$. In particular, if ω denotes the angle of the corner, then $\lambda_1(\omega)$ is the first non-zero eigenvalue of the operator $\mathcal{L}$. In these works, only the regularity of the solution has been studied without much details on the specific expressions of the singularities of the solution. Although the solution is not regular enough when the right-hand side is smooth, explicit expressions of the the singularities of the solution are interesting and helpful in solving the equations numerically [1, 2, 18].

Under general boundary conditions, a decomposition of the solution in the form of

$$(\mathbf{u}, p) = (\mathbf{u}_s, p_s) + \sum_{j,k} C_{\lambda_j,k} (\ln r)^k (r^{\lambda_j} \mathbf{u}_{\lambda_j,\omega,k}(\theta), r^{\lambda_j - 1} p_{\lambda_j,\omega,k}(\theta)) \tag{5}$$

has been mentioned in [17], where the first part is regular and the second part contains the singularities. Under the Dirichlet boundary condition and the condition $g = 0$, the Stokes problem can be converted to a fourth-order bihamornic equation and by this method one can show that only first-order logarithmic singularities appear in the above expression [9], but this method does not apply to the inhomogeneous stress/mixed boundary condition or the case $g \neq 0$. Recently, Choi and