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Multi-Scale Deep Neural Network (MscaleDNN) Methods for Oscillatory Stokes Flows in Complex Domains

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Summary. In this paper, we study a multi-scale deep neural network (MscaleDNN) as a meshless numerical method for computing oscillatory Stokes flows in complex domains. The MscaleDNN employs a multi-scale structure in the design of its DNN using radial scalings to convert the approximation of high frequency components of the highly oscillatory Stokes solution to one of lower frequencies. The MscaleDNN solution to the Stokes problem is obtained by minimizing a loss function in terms of L^2 norm of the residual of the Stokes equation. Three forms of loss functions are investigated based on vorticity-velocity-pressure, velocity-stress-pressure, and velocity-gradient of velocity-pressure formulations of the Stokes equation. We first conduct a systematic study of the MscaleDNN methods with various loss functions on the Kovasznay flow in comparison with normal fully connected DNNs. Then, Stokes flows with highly oscillatory solutions in a 2-D domain with six randomly placed holes are simulated by the MscaleDNN. The results show that MscaleDNN has faster convergence and consistent error decays in the simulation of Kovasznay flow for all three tested loss functions. More importantly, the MscaleDNN is capable of learning highly oscillatory solutions when the normal DNNs fail to converge.

AMS subject classifications: 35Q68, 65N99, 68T07, 76M99

Key words: Deep neural network, Stokes equation, multi-scale, meshless methods.

1 Introduction

Numerical methods for incompressible flow is one of the major topics in computational fluid dynamics, which has been intensively studied over last five decades. Various tech-

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niques have been proposed to address the incompressibility condition of the flow, including projection methods [4, 18], Gauge methods [6], and time splitting methods [13], among others. Finite element and spectral element methods [3] are often used to discretize the Navier-stokes equation where special attentions are needed for the approximation spaces of velocity and pressure to satisfy the Babuska and Brezzi inf-sup condition for a saddle point problem [8]. Besides, for large scale engineering applications, body-fitted mesh generations for 3-D objects and efficient linear solvers for the resulting linear systems have been a major issue for computational resources.

The emerging deep neural network (DNN) has found many applications beyond its traditional applications such as image classification and speech recognition. Recent work in extending DNNs to the field of scientific and engineering computing has shown much promise [7, 9, 17]. DNN based numerical methods are usually formulated as an optimization problem where the loss function could be an energy functional as in a Ritz formulation of a self-adjoint differential equation [7] or simply the least squared mean of the residual of the PDEs [2, 10, 11]. The DNN technique provides a powerful approximation method to represent solutions of high dimensional variables while the traditional finite element and spectral element methods encounter the well known curse of dimensionality problem. Also, there are several advantages of using DNN to approximate the solution of the incompressible flows. Firstly, the stochastic optimization algorithm employed by DNN based methods relies on loss calculated on randomly sampled points in the computational domain rather than over an unstructured mesh fitting the geometry of the complex objects in the fluid problem. This feature renders the DNN-based methods for solving PDEs a truly mesh-less method. Secondly, due to the capability of the DNN in handling high dimensional functions, the approximation of a time dependent solution can be carried out in the temporal-spatial four dimensional space. Thirdly, boundary conditions for the fluid problems can be simply enforced by introducing penalty terms in the loss function and no need to find and implement appropriate and non-trivial boundary conditions for pressure [16] or vorticity variables in corresponding formulations for the Stokes or Navier-Stokes equations.

Normal fully connected DNNs used for image classification and data science applications have been shown to be ineffective in learning high frequency contents of the solution as illustrated in recent works on DNNs' frequency dependent convergence properties [19]. Unfortunately, fluid flow at high Reynolds number will contain many scales, which is the hallmark of the onset of a turbulent flow from a laminar one. Therefore, in order to make the DNN based approaches to be competitive numerical methods, in terms of resolution power, to popular spectral [3] and spectral element methods [12], it is important to develop new classes of DNNs which can represent scales of drastic disparities arising from the study of turbulent flows. For this purpose, we have recently developed strategies to speed up the convergence of DNNs in learning high frequency content of the solutions of PDEs. Two new DNNs have been proposed: a PhaseDNN [2] and a MscaleDNN [11]. The PhaseDNN uses a series of phase shifts to convert high frequency contents to a low frequency range before the learning is carried out. This method