Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2019-0197

Vol. **29**, No. 1, pp. 237-264
January 2021

A Kernel Based Unconditionally Stable Scheme for Nonlinear Parabolic Partial Differential Equations

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Received 6 November 2019; Accepted (in revised version) 28 May 2020

Abstract. In this paper, a class of high order numerical schemes is proposed to solve the nonlinear parabolic equations with variable coefficients. This method is based on our previous work [11] for convection-diffusion equations, which relies on a special kernel-based formulation of the solutions and successive convolution. However, disadvantages appear when we extend the previous method to our equations, such as inefficient choice of parameters and unprovable stability for high-dimensional problems. To overcome these difficulties, a new kernel-based formulation is designed to approach the spatial derivatives. It maintains the good properties of the original one, including the high order accuracy and unconditionally stable for one-dimensional problems, hence allowing much larger time step evolution compared with other explicit schemes. In additional, without extra computational cost, the proposed scheme can enlarge the available interval of the special parameter in the formulation, leading to less errors and higher efficiency. Moreover, theoretical investigations indicate that it is unconditionally stable for multi-dimensional problems as well. We present numerical tests for one- and two-dimensional scalar and system, demonstrating the designed high order accuracy and unconditionally stable property of the scheme.

AMS subject classifications: 65M12

Key words: Nonlinear parabolic equation, kernel based scheme, unconditionally stable, high order accuracy.

1 Introduction

In this work, we want to solve the nonlinear parabolic equations

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$$\partial_t u(\mathbf{x},t) = \nabla\cdot(\mathbf{A}(u,\mathbf{x},t)\nabla u) + \mathbf{B}(u,\mathbf{x},t)^T\nabla u + C(u,\mathbf{x},t), \tag{1.1}$$

on the domain $\mathbf{x}\in\Omega\subset\mathbb{R}^n$ with initial and boundary conditions. Here, $\mathbf{A}(u,\mathbf{x},t) = \big(A_{ij}(u,\mathbf{x},t)\big)\in\mathbb{R}^{n\times n}$ and $\mathbf{B}(u,\mathbf{x},t) = (B_i(u,\mathbf{x},t))\in\mathbb{R}^n$. In particular, Eq. (1.1) is parabolic if there exists a constant $\theta>0$ such that

$$\sum_{i,j=1}^{n} A_{ij}\xi_i\xi_j \geq \theta\sum_{i=1}^{n}\xi_i^2, \qquad \forall(\xi_1,\cdots,\xi_n)\in\mathbb{R}^n.$$

For such a time-dependent partial differential equation (PDE) (1.1), one common method is splitting the equation into a system with an auxiliary variable $\mathbf{w}\in\mathbb{R}^n$ at first,

$$\begin{cases} \partial_t u(\mathbf{x},t) = \nabla\cdot(\mathbf{A}(u,\mathbf{x},t)\mathbf{w}) + \mathbf{B}(u,\mathbf{x},t)^T\nabla u + C(u,\mathbf{x},t), \\ \mathbf{w} = \nabla u. \end{cases} \tag{1.2}$$

And then solve the two equations at the same time level. There is a large amount of numerical methods for this problem. Most of these schemes discretize the spatial variables at first with finite volume / difference methods, finite element methods, or spectral methods, generating a large coupled system of ordinary differential equations (ODEs). And then apply an initial value ODE solver in time. This approach is commonly referred to as the Method of Lines (MOL) and interested readers are referred to [29] for further discussions. Classical methods for this time evolution include multi-step, multi-stage, or multi-derivative methods, as well as a combination of these approaches. For instance, the Runge-Kutta method and the Taylor series methods. Note that efficiency is a main concern of these schemes. For example, the explicit methods solving (1.2) does restrict the time step $\Delta t\propto\Delta x^2$ due to the stability requirement, where Δx is the spatial mesh size. Using Implicit-Explicit (IMEX) or fully implicit time discretization techniques [23,28] can allow larger time step, but usually we need to solve a system of (nonlinear) equations for each step. The algorithm would be expensive when the system size becomes bigger. Besides the classical ones, other high order time discretization techniques were also developed, e.g., the spectral deferred correction (SDC) method [15,19,26], the exponential time differencing method [14,22], the integration factor methods [2,14,21,25,27], and the hybrid methods of SDC and high order Runge-Kutta schemes [12].

Another framework named the Method of Lines Transpose (MOL^T) has been exploited in the literature for solving the linear time-dependent PDEs. In such a framework, the temporal variable is first discretized, resulting in a set of linear boundary value problems (BVPs) at discrete time levels. Furthermore, each BVP can be inverted analytically in an integral formulation based on a kernel function and then the numerical solution is updated accordingly. As a notable advantage, the MOL^T approach is able to use an implicit method but avoid solving linear systems at each time step, see [5]. Moreover, a fast convolution algorithm is developed to ensure the computational complexity of the scheme is $\mathcal{O}(N)$ [1,7,18], where N is the number of discrete mesh points.