Ann. of Appl. Math.
36:4(2020), 331-355

INVERSE SCATTERING METHOD AND APPLICATIONS *

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Abstract

We briefly introduce the applications of the inverse scattering method to some particular class of problems, which yield the exact solution or long time asymptotics in terms of the unique solution of a matrix Riemann-Hilbert problem formulated in the complex plane.

Keywords inverse scattering; Riemann-Hilbert problem; long-time asymptotics; initial value problem (IVP); initial boundary value problem (IBVP)

2000 Mathematics Subject Classification 35Q15; 37K40; 37K15; 14K25; 35Q60

1 Introduction

In 1967, Gardner, Greene, Kruskal and Miura (GGKM) proposed the inverse scattering method. It's the most effective method for the exact solution of the IVPs for the nonlinear evolution equations whose initial values decay sufficiently rapidly, through a series of linear equations rather than by linearized perturbation method. We will obtain the more direct and clearer picture of the solution by the inverse scattering method, which is different from the conventional energy method of nonlinear partial differential equations, for example, the method helps to get the behaviors of $t \to \infty$ or $x \to \pm\infty$ for the Korteweg-de Vries (KdV) equation, the nonlinear Schrödinger equation (NLS) and so on. Meanwhile, the inverse scattering method as a new method is used to prove the regularity of the solution. Many scholars have done a lot of excellent work in this field in recent years [13–21]. This paper briefly introduces the research status and recent progress in this field.

2 The Inverse Scattering Method and the IVP for the KdV Equation

In this section, we briefly introduce the inverse scattering method for the exact solution of the IVP for the KdV equation.

*Manuscript received September 28, 2020
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For the Burgers equation

$$u_t + uu_x - \alpha u_{xx} = 0, \quad \alpha > 0, \tag{2.1}$$

using the Hopf-Cole transformation

$$u = -2\alpha \frac{w_x}{w}, \tag{2.2}$$

it can be converted to a linear thermal conductivity equation about w

$$w_t = \alpha w_{xx}, \tag{2.3}$$

and the original equation has the solution

$$u(x,t) = \frac{\displaystyle\int_{-\infty}^{\infty} \frac{x-\xi}{t} \exp\Big[-\frac{(x-\xi)^2}{4\alpha t} - \frac{1}{2\alpha}\int_0^{\xi} u_0(\xi')\mathrm{d}\xi' \Big] \mathrm{d}\xi}{\displaystyle\int_{-\infty}^{\infty} \exp\Big[-\frac{(x-\xi)^2}{4\alpha t} - \frac{1}{2\alpha}\int_0^{\xi} u_0(\xi')\mathrm{d}\xi' \Big] \mathrm{d}\xi}, \tag{2.4}$$

where $u_0(x)$ is the initial condition, $u|_{t=0} = u_0(x)$.

In 1952, E. Hopf proved that (2.4) tends to the generalized solution of quasilinear hyperbolic equation ($u_t + uu_x = 0$) as $\alpha \to 0$, and the basic theory of quasilinear hyperbolic type theory was established accordingly. For other nonlinear partial differential equations, it's hard to reduce them to linear partial differential equations.

In 1967, GGKM found a new precise method for the exact solution of the IVP for the complete integrable KdV equation through a series of linear equations. Consider the KdV equation

$$u_t - 6uu_x + u_{xxx} = 0. \tag{2.5}$$

Taking $u = v^2 + v_x + \lambda$, (2.5) can be viewed as a Riccati equation for v in terms of u; taking further $v = \frac{\psi_x}{\psi}$, which yields 1d Schrödinger equation

$$\psi_{xx} - (u - \lambda)\psi = 0 \tag{2.6}$$

where ψ is the wave function, u is the potential, and λ is the energy spectrum.

If the solution of KdV equation is smooth and bounded, there is $u \to 0$ as $|x| \to \infty$. Then the spectrum of the Schrödinger equation (2.6) consists of a finite number of discrete eigenvalues $\lambda_m = -k_m^2$ ($m = 1, 2, \cdots, N$) for $\lambda < 0$ and a continuum $\lambda = k^2$ ($-\infty < k < \infty$, k is real) for $\lambda > 0$.

Fixed t, define the solution of scattering problem (2.6) satisfying the boundary conditions (shown in Figure 1)

$$\begin{cases} \psi(x,k,t) \sim \mathrm{e}^{-ikx} + b(k,t)\mathrm{e}^{ikx}, & x \to +\infty; \\ \psi(x,k,t) \sim a(k,t)\mathrm{e}^{-ikx}, & x \to -\infty; \end{cases} \tag{2.7}$$

and

$$\begin{cases} \psi_m(x,k_m(t),t) \sim C_m(k_m(t),t)\mathrm{e}^{-k_m x}, & x \to +\infty; \\ \psi_m(x,k_m(t),t) \sim \mathrm{e}^{k_m x} & x \to -\infty, \end{cases} \tag{2.8}$$