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POSITIVE SOLUTIONS TO A BVP WITH TWO INTEGRAL BOUNDARY CONDITIONS*†

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Abstract

Based on the Guo-Krasnoselskii's fixed-point theorem, the existence and multiplicity of positive solutions to a boundary value problem (BVP) with two integral boundary conditions

$$\begin{cases} v^{(4)} = f(s, v(s), v'(s), v''(s)), \quad s \in [0,1], \\ v'(1) = v'''(1) = 0, \\ v(0) = \displaystyle\int_0^1 g_1(\tau)v(\tau)\mathrm{d}\tau, \quad v''(0) = \displaystyle\int_0^1 g_2(\tau)v''(\tau)\mathrm{d}\tau \end{cases}$$

are obtained, where f, g_1, g_2 are all continuous. It generalizes the results of one positive solution to multiplicity and improves some results for integral BVPs. Moreover, some examples are also included to demonstrate our results as applications.

Keywords integral boundary conditions; positive solutions; cone
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1 Introduction

Boundary value problems (BVPs) for ordinary differential equations have wide applications in many scientific areas such as physics, mechanics of materials, ecology and so on. For example, deformations of elastic beams can be represented for some fourth-order BVPs, and there are some appealing results [1–3] can be referred.

Especially, much attention has been drawn to BVPs with integral boundary conditions [4–9] recent years because of their applications in thermodynamics and chemical engineering. In 2011, by global bifurcation theory and the Krein-Rutman theorem, Ma [4] investigated positive solutions to a class of BVP with integral boundary conditions. Hereafter, based on the famous Krasnoselskii's fixed-point theorem,

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Lv [5] considered the monotone and concave positive solutions to the following integral BVP

$$\begin{cases} v^{(4)}(s) = f(s, v(s), v'(s), v''(s)), \quad s \in [0,1], \\ v(0) = v'(1) = v'''(1) = 0, \\ v''(0) = \displaystyle\int_0^1 g(\tau)v''(\tau)\mathrm{d}\tau, \end{cases} \tag{1.1}$$

where $f \in C([0,1] \times [0,+\infty) \times [0,+\infty) \times (-\infty,0], [0,+\infty))$, $g \in C([0,1],[0,+\infty))$. For more this kind of results, one can refer to [7–9].

Nevertheless, there are few results on multiplicity and the properties of positive solutions to BVPs with integral boundary conditions. Recently, Yang [6] investigated the multiplicity of positive solutions to a fourth-order integral BVP. Motivated by Yang's ideas in [6] and based on the work of Lv [5], this paper deals with the following BVP with two integral boundary conditions

$$\begin{cases} v^{(4)}(s) = f(s, v(s), v'(s), v''(s)), \quad s \in [0,1], \\ v'(1) = v'''(1) = 0, \\ v(0) = \displaystyle\int_0^1 g_1(\tau)v(\tau)\mathrm{d}\tau, \quad v''(0) = \displaystyle\int_0^1 g_2(\tau)v''(\tau)\mathrm{d}\tau, \end{cases} \tag{1.2}$$

and the existence and multiplicity of positive solutions are thus established. Notice that (1.2) is reduced to (1.1) when $g_1 \equiv 0$ in (1.2), and therefore we generalize the result of one positive solution in [5] to the case of multiple positive solutions. Moreover, some results of positive solutions to integral BVPs we mentioned in [4,5,9] are also improved.

2 Preliminaries

We first state several notations and lemmas in this paper.

We always assume that, throughout this paper, $f : [0,1] \times [0,+\infty) \times [0,+\infty) \times (-\infty,0] \to [0,+\infty)$ and $g_1, g_2 : [0,1] \to [0,+\infty)$ are all continuous. Furthermore,

$$\zeta_1 := \int_0^1 g_1(\tau)\mathrm{d}\tau < \frac{1}{2}, \quad \zeta_2 := \int_0^1 g_2(\tau)\mathrm{d}\tau < 1 \quad \text{and} \quad g_1 \le \frac{1-2\zeta_1}{2(1-\zeta_2)} g_2.$$

Denote

$$\overline{f_0} = \limsup_{v_0+v_1-v_2 \to 0^+} \max_{s\in[0,1]} \frac{f(s,v_0,v_1,v_2)}{v_0+v_1-v_2}, \quad \overline{f_\infty} = \limsup_{v_0+v_1-v_2 \to +\infty} \max_{s\in[0,1]} \frac{f(s,v_0,v_1,v_2)}{v_0+v_1-v_2},$$

$$\underline{f_0} = \liminf_{v_0+v_1-v_2 \to 0^+} \min_{s\in[0,1]} \frac{f(s,v_0,v_1,v_2)}{v_0+v_1-v_2}, \quad \underline{f_\infty} = \liminf_{v_0+v_1-v_2 \to +\infty} \min_{s\in[0,1]} \frac{f(s,v_0,v_1,v_2)}{v_0+v_1-v_2},$$

$$M_1 = \frac{3}{2(1-\zeta_2)}, \quad M_2 = \frac{1}{4}\int_0^1 \tau^2\Big(1-\frac{1}{2}\tau\Big)\Big[\frac{1}{2} + \frac{1}{1-\zeta_2}\int_0^1 s g_2(s)\mathrm{d}s\Big]\mathrm{d}\tau.$$