Journal of Nonlinear Modeling and Analysis
Volume 2, Number 1, March 2020, 57–78
http://jnma.ca; http://jnma.ijournal.cn
DOI:10.12150/jnma.2020.57

Bi-center Problem in a Class of Z_2-equivariant Quintic Vector Fields*

Hongwei Li[1], Feng Li[1,†] and Pei Yu[2]

Abstract In this paper, we study the center problem for Z_2-equivariant quintic vector fields. First of all, for convenience in analysis, the system is simplified by using some transformations. When the system has two nilpotent points at $(0, \pm 1)$ with multiplicity three, the first seven Lyapunov constants at the singular points are calculated by applying the inverse integrating factor method. Then, fifteen center conditions are obtained for the two nilpotent singular points of the system to be centers, and the sufficiency of the first seven center conditions are proved. Finally, the first five Lyapunov constants are calculated at the two nilpotent points $(0, \pm 1)$ with multiplicity five by using the method of normal forms, and the center problem of this system is partially solved.

Keywords Nilpotent singular point, Center–focus problem, Bi-center, Lyapunov constant.

MSC(2010) 34C05, 34C07.

1. Introduction

Consider the following planar differential system,

$$\frac{dx}{dt} = P(x, y), \quad \frac{dy}{dt} = Q(x, y), \tag{1.1}$$

where P and Q are polynomials. The second part of Hilbert's 16th problem is to find the upper bound of the limit cycles that system (1.1) can have. One important problem related to the bifurcation of limit cycles is to determine whether a singular point of system (1.1) is a center or not, which is called center problem. The distinction between a center case and a non-center case has great difference on the determination of limit cycles.

As a special class of system (1.1), the Z_n-equivariant vector fields have attractive properties because of their symmetry. It is well known that better results on the number of limit cycles are often obtained from Z_n-equivariant vector fields. In

[†]the corresponding author.
Email address:lf0539@126.com(Feng Li)
[1]School of Mathematics and Statistics, Linyi University, Linyi, Shandong 276005, China
[2]Department of Applied Mathematics, Western University, London, Ontario N6A 5B7, Canada
*This research was partially supported by the National Natural Science Foundation of China, No. 11601212 (F. Li), Shandong Natural Science Foundation, No. ZR2018MA002 (F. Li) and the Natural Sciences and Engineering Research Council of Canada, No. R2686A02 (P. Yu).

recent years, more and more attention has been paid to the center problem of Z_n-equivariant vector fields. For example, Liu and Li [1] gave a complete study on the bi-center problem of a class of Z_2-equivariant cubic vector fields. Romanovski *et al.* [2] studied the bi-center problem of some Z_2-equivariant quintic systems. Giné [3] investigated the coexistence of centres in two families of planar Z_n-equivariant systems. Theory of rotated equations was discussed by Han *et al.* [4] and applied to study a population model. The Poincaré return map and generalized focal values of analytic planar systems with a nilpotent focus or center were considered in [5] where the classical Hopf bifurcation theory was generalized. Global phase portraits of symmetrical cubic Hamiltonian systems with a nilpotent singular point were discussed in [6]. Recently, Yu *el al.* [7] applied the method of normal forms to improve the results on the number of limit cycles bifurcating from a non-degenerate center of various homogeneous polynomial differential systems. A special type of bifurcation of limit cycles from a nilpotent critical point was studied in [8]. However, for degenerate singular points, because of difficulty, there are very few results obtained even for Z_2-equivalent systems with two nilpotent singular points. In [9], the authors proved that the origin of any Z_2-symmetric system is a nilpotent center if and only if there exists a local analytic first integral. Recently, we studied bifurcation of limit cycles in a class of Z_2-equivalent cubic planar differential systems with two nilpotent singular points, described by

$$\begin{aligned}\frac{dx}{dt} &= A_{10}x + A_{01}y + A_{30}x^3 + A_{21}x^2y + A_{12}xy^2 + A_{03}y^3 = X(x,y),\\ \frac{dy}{dt} &= B_{10}x + B_{01}y + B_{30}x^3 + B_{21}x^2y + B_{12}xy^2 + B_{03}y^3 = Y(x,y).\end{aligned} \tag{1.2}$$

In [10], sufficient and necessary conditions for the critical points of the system (1.2) to be centers were obtained. In addition, the existence of 12 small-amplitude limit cycles bifurcating from the critical points was proved.

In this paper, a class of Z_2-equivariant quintic planar differential systems with two nilpotent singular points, given by

$$\begin{aligned}\frac{dx}{dt} =&A_{10}x + A_{01}y + A_{50}x^5 + A_{41}x^4y + A_{32}x^3y^2 + A_{23}x^2y^3\\ &+ A_{14}xy^4 + A_{05}y^5 = X(x,y),\\ \frac{dy}{dt} =&B_{10}x + B_{01}y + B_{50}x^5 + B_{41}x^4y + B_{32}x^3y^2 + B_{23}x^2y^3\\ &+ B_{14}xy^4 + B_{05}y^5 = Y(x,y),\end{aligned} \tag{1.3}$$

are studied. Necessary conditions for the singular points of system (1.3) to be centers are derived.

The rest of the paper is organized as follows. In the next section, we simplify system (1.3) for convenience in analysis. In Section 3, the first seven Lyapunov constants at an order-3 nilpotent singular point are computed by using the inverse integrating factor method or the method of normal forms. Bi-center conditions in Z_2-equivariant vector fields are discussed, and fifteen bi-center conditions are obtained for system (1.3). Further, the first five Lyapunov constants at an order-5 nilpotent singular point are computed by using the method of normal forms, yielding one more bi-center condition.