Journal of Nonlinear Modeling and Analysis http://jnma.ca; http://jnma.ijournal.cn
Volume 2, Number 1, March 2020, 95–110 DOI:10.12150/jnma.2020.95

Applications of Variational Method to Impulsive Fractional Differential Equations with Two Control Parameters*

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Abstract In this paper, we study the existence of the impulsive fractional differential equation. Based on a previous paper [2], we give more accurate condition to guarantee the impulsive fractional differential equation has at least three solutions under certain assumptions by using variational methods and critical point theory. Moreover, some recent results are generalized and significantly improved.

Keywords Impulsive fractional differential equations, Variational methods, Critical point theory, Three solutions.

MSC(2010) 34A60, 34B37, 58E05.

1. Introduction

In this paper, we will consider the following fractional differential equation with impulsive effects

$$\begin{cases} \frac{d}{dt}\{ {}_0\mathrm{D}_t^{\alpha-1}({}^c{}_0\mathrm{D}_t^{\alpha}u(t)) - {}_t\mathrm{D}_T^{\alpha-1}({}^c{}_t\mathrm{D}_T^{\alpha}u(t))\} \\ +\lambda f(t,u(t)) + \mu g(t,u(t)) = 0, t \in [0,T], t \neq t_k, \\ \Delta(\ \mathrm{D}_t^{\alpha}u)(t_k) = I_k(u(t_k)), t = t_k, k = 1,2,...l, \\ u(0) = u(T) = 0, \end{cases} \tag{1.1}$$

where $\alpha \in (\frac{1}{2}, 1]$, ${}_0\mathrm{D}_t^{\alpha-1}$ and ${}_t\mathrm{D}_T^{\alpha-1}$ represent the left and right Riemann-Liouville fractional integrals of order $1-\alpha$, ${}^c{}_0\mathrm{D}_t^{\alpha}$ and ${}^c{}_t\mathrm{D}_T^{\alpha}$ represent the left and right Caputo fractional derivative of order α, respectively. $f, g : [0,T] \times R \to R$ are given continuous functions, λ and μ are positive parameters, $I_k : R \to R$, $k = 1,2,...l$ are continuous functions and

$$(\ \mathrm{D}_t^{\alpha}u)(t) = \Big\{ {}_0\mathrm{D}_t^{\alpha-1}({}^c{}_0\mathrm{D}_t^{\alpha}u) - {}_t\mathrm{D}_T^{\alpha-1}({}^c{}_t\mathrm{D}_T^{\alpha}u) \Big\}(t),$$

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*The authors were supported by the NSFC(11571088, 11471109), the Zhejiang Provincial Natural Science Foundation of China (LY14A010024), and Scientific Research Fund of Hunan Provincial Education Department (14A098).

$$\Delta(\ \mathrm{D}_t^{\alpha}u)(t_k) = \Big\{\ {}_0\mathrm{D}_t^{\alpha-1}({}^c{}_0\mathrm{D}_t^{\alpha}u) -\ {}_t\mathrm{D}_T^{\alpha-1}({}^c{}_t\mathrm{D}_T^{\alpha}u)\Big\}(t_k^+) \\ - \Big\{\ {}_0\mathrm{D}_t^{\alpha-1}({}^c{}_0\mathrm{D}_t^{\alpha}u) -\ {}_t\mathrm{D}_T^{\alpha-1}({}^c{}_t\mathrm{D}_T^{\alpha}u)\Big\}(t_k^-),$$

$$\Big\{\ {}_0\mathrm{D}_t^{\alpha-1}({}^c{}_0\mathrm{D}_t^{\alpha}u)-\ {}_t\mathrm{D}_T^{\alpha-1}({}^c{}_t\mathrm{D}_T^{\alpha}u)\Big\}(t_k^+) = \lim_{t\to t_k^+}\Big\{\ {}_0\mathrm{D}_t^{\alpha-1}({}^c{}_0\mathrm{D}_t^{\alpha}u)-\ {}_t\mathrm{D}_T^{\alpha-1}({}^c{}_t\mathrm{D}_T^{\alpha}u)\Big\}(t_k),$$

$$\Big\{\ {}_0\mathrm{D}_t^{\alpha-1}({}^c{}_0\mathrm{D}_t^{\alpha}u)-\ {}_t\mathrm{D}_T^{\alpha-1}({}^c{}_t\mathrm{D}_T^{\alpha}u)\Big\}(t_k^-) = \lim_{t\to t_k^-}\Big\{\ {}_0\mathrm{D}_t^{\alpha-1}({}^c{}_0\mathrm{D}_t^{\alpha}u)-\ {}_t\mathrm{D}_T^{\alpha-1}({}^c{}_t\mathrm{D}_T^{\alpha}u)\Big\}(t_k),$$

for $k = 1, \cdots, l$.

In recent years, more and more attention have been paid to the fractional differential equations have obtained by many authors. By using variational methods and some critical point theory, some interesting results on fractional differential equations which have been presented to our vision, see [2–16] and the references therein.

More precisely, in a recent paper [2], the authors have considered the following fractional boundary problem without impulsive effects

$$\begin{cases} \frac{d}{dt}\{\ {}_0\mathrm{D}_t^{\alpha-1}({}^c{}_0\mathrm{D}_t^{\alpha}u(t)) -\ {}_t\mathrm{D}_T^{\alpha-1}({}^c{}_t\mathrm{D}_T^{\alpha}u(t))\} \\ +\lambda f(t,u(t)) + \mu g(t,u(t)) = 0, t\in[0,T], \\ u(0) = u(T) = 0, \end{cases} \tag{1.2}$$

the main result is as follows:

Theorem 1.1. *[Theorem 3.1, [2]] Assume that there exist positive constants c, d with*

$$c < (\frac{4d\Omega}{T\Gamma(2-\alpha)})\sqrt{C(T,\alpha)}, \tag{1.3}$$

such that

(A1) $F(t,\xi)\ge 0$, for each $(t,\xi)\in([0,\frac{T}{4}]\bigcup[\frac{3T}{4},T])\times[0,d]$;

(A2) $\frac{\int_0^T \max_{|\xi|\le c} F(t,\xi)dt}{c^2} < \frac{|\cos(\pi\alpha)|\int_{\frac{T}{4}}^{\frac{3T}{4}} F(t,d)dt}{\Omega^2\omega_{\alpha,d}}$;

(A3) $\limsup_{|\xi|\to+\infty}\frac{\sup_{t\in[0,T]}F(t,\xi)}{\xi^2} < \frac{\int_0^T \max_{|\xi|\le c}F(t,\xi)dt}{2c^2T}$.

Then, for every $\lambda\in\Lambda$ and for every continuous function $g:[0,T]\times R\to R$ such that

$$\limsup_{|\xi|\to+\infty}\frac{\sup_{t\in[0,T]}G(t,\xi)}{\xi^2} < +\infty,$$

where $F(t,\xi) = \int_0^{\xi} f(t,s)ds$ and $G(t,\xi) = \int_0^{\xi} f(t,s)ds$. Then there exists $\bar{\delta}$ such that for each $\mu\in[0,\bar{\delta}]$, problem (1.2) admits at least three solutions.

In fact, Theorem 1.1 is not valid. In [2], the authors fixed $c, d > 0$ such that

$$\frac{\omega_{\alpha,d}}{\int_{\frac{T}{4}}^{\frac{3T}{4}} F(t,d)dt} < \frac{\frac{|\cos(\pi\alpha)|}{\Omega_2^2}c^2}{\int_0^T \max_{|u|\le c}F(t,u)dt}, \tag{1.4}$$

holds, where

$$\omega_{\alpha,d} = \frac{16d^2}{T^2\Gamma^2(2-\alpha)|\cos(\pi\alpha)|}C(T,\alpha)$$