Journal of Nonlinear Modeling and Analysis http://jnma.ca; http://jnma.ijournal.cn
Volume 2, Number 3, September 2020, 431–446 DOI:10.12150/jnma.2020.431

Complete Hyper-elliptic Integrals of the First Kind and the Chebyshev Property*

Jihua Yang[1,†]

Abstract This paper is devoted to study the following complete hyper-elliptic integral of the first kind

$$J(h) = \oint_{\Gamma_h} \frac{\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3}{y} dx,$$

where $\alpha_i \in \mathbb{R}$, Γ_h is an oval contained in the level set $\{H(x, y) = h, h \in (-\frac{5}{36}, 0)\}$ and $H(x, y) = \frac{1}{2}y^2 - \frac{1}{4}x^4 + \frac{1}{9}x^9$. We show that the 3-dimensional real vector spaces of these integrals are Chebyshev for $\alpha_0 = 0$ and Chebyshev with accuracy one for $\alpha_i = 0$ $(i = 1, 2, 3)$.

Keywords Complete hyper-elliptic integral of the first kind, Chebyshev, ECT-system.

MSC(2010) 34C07, 34C05.

1. Introduction and main results

In 1990, Arnold [1] proposed ten problems among which the 7th problem is on the number of zeros of Abelian integrals, which can be stated in the following way: consider the Abelian integral

$$I(h) = \oint_{\Gamma_h} P(x, y)dy + Q(x, y)dx, \quad h \in \mathbb{J},$$

where Γ_h is a family of closed curves of a real polynomial $H(x, y) = h$, $P(x, y)$, $Q(x, y)$ and $H(x, y)$ are polynomials satisfying $\max\{\deg P, \deg Q\} = n$ and $\deg\{H\} = m + 1$, $\mathbb{J}$ is an open interval. How large can the number of isolated zeros of the function $I(h)$ in the open interval $\mathbb{J}$? And for the complete hyper-elliptic integral of the first kind

$$J(h) = \oint_{\Gamma_h} \frac{\alpha_0 + \alpha_1 x + \cdots + \alpha_{g-1} x^{g-1}}{y} dx, \; H(x, y) = y^2 + U(x),$$

where $\deg U = 2g + 1 > 4$, α_i $(i = 1, 2, \cdots g - 1)$ are real parameters. Is the g-dimensional family of $J(h)$ a Chebyshev family in the open interval? Where

† the corresponding author.
Email address: yangjh@mail.bnu.edu.cn(J. Yang)
1 School of Mathematics and Computer Science, Ningxia Normal University, Guyuan, Ningxia 756000, China
* The author was National Natural Science Foundation of China (11701306), Young Top-notch Talent of Ningxia, Ningxia Natural Science Foundation(2019AAC03247) and Construction of First-class Disciplines of Higher Education of Ningxia(pedagogy)(NXYLXK2017B11).

Chebyshev family means that the number of the isolated zeros of $J(h)$ is smaller than $g-1$.

It is known to all that the first part of the 7th problem is so-called the weakened 16th Hilbert problem compare to Hilbert in [13]. On the theme there have been many excellent works, see [2–6,10,11,14–18,20,21,23–28] and the references therein.

However, there are few works on the second part of the 7th problem, especially for $g>2$. Gavrilov and Iliev [8] obtained that the g-dimensional real vector space of $J(h)$ is not Chebyshev for any $g>1$, and when g $=2$ and deg $U=5$ there exist exceptional families of ovals $\{\Gamma_h\}$ of $y^2+U(x)=h$ such that every Abelian integral of the form

$$J(h)=\oint_{\Gamma_h}\frac{\alpha_0+\alpha_1 x}{y}dx,\quad \alpha_0^2+\alpha_1^2\neq 0$$

has at most one isolated zero for h in an open interval $\mathbb{I}$. Wang, Wang and Xiao [22] studied the Chebyshev property of the above $J(h)$ for three classes of degenerate families of ovals Γ_h in [8]. It is shown that the three classes of complete hyper-elliptic integrals are Chebyshev, and the exact bounds on the number of zeros of these Abelian integrals are one.

In this paper, motivated by the above results, especially by [1,8,22], we investigate the following hyper-elliptic Hamilton system

$$\dot{x}=y,\quad \dot{y}=-x^3(x^5-1), \tag{1.1}$$

whose Hamiltonian is

$$H(x,y)=\frac{1}{2}y^2-\frac{1}{4}x^4+\frac{1}{9}x^9:=\frac{1}{2}y^2+U(x). \tag{1.2}$$

The oval $H(x,y)=-\frac{5}{36}$ corresponds to the center $C(1,0)$, the oval $H(x,y)=0$ corresponds to the homoclinic through the nilpotent saddle point $O(0,0)$, see Figure 1. It intersects the positive x-axis at point $(\frac{1}{2}\sqrt[5]{72},0)$. The corresponding complete hyper-elliptic integral of the first kind is

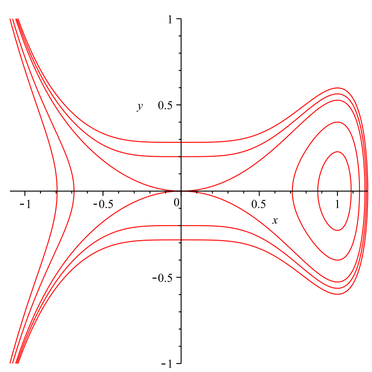

Figure 1. The level curves of $H(x,y)=h$.

$$\begin{aligned}\mathcal{J}(h)=&\oint_{\Gamma_h}\frac{\alpha_0+\alpha_1 x+\alpha_2 x^2+\alpha_3 x^3}{y}dx\\ :=&\alpha_0\mathcal{J}_0(h)+\alpha_1\mathcal{J}_1(h)+\alpha_2\mathcal{J}_2(h)+\alpha_3\mathcal{J}_3(h),\end{aligned}$$