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Crank-Nicolson Method of a Two-Grid Finite Volume Element Algorithm for Nonlinear Parabolic Equations

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Abstract. A two-grid finite volume element algorithm based on Crank-Nicolson scheme for nonlinear parabolic equations is proposed. In this method, the nonlinear problem is solved on a coarse grid of size H and a linear problem is considered on a fine grid of size h by using the coarse-grid solution and one Newton iteration. This helps to improve the computing efficiency while keeping the accuracy. It is proved that the two-grid method can achieve asymptotically optimal error estimates in spaces and second order accuracy in time. Numerical results are consistent with the theoretical findings.

AMS subject classifications: 65N12, 65M60

Key words: Crank-Nicolson method, two-grid algorithm, finite volume element method, error estimates, nonlinear parabolic equations.

1. Introduction

We consider the following nonlinear parabolic problem:

$$
\begin{aligned}
&u_t - \nabla\cdot\big(d(X,t)\nabla u\big) = f(u), && (X,t)\in\Omega\times J,\\
&u(X,t) = 0, && (X,t)\in\partial\Omega\times J,\\
&u(X,0) = u_0(X), && X\in\Omega,
\end{aligned}
\tag{1.1}
$$

where $\Omega\subset\mathbb{R}^2$ is a convex polygonal domain, $X=(x,y)$, $J=[0,T]$, and $f(u)=f(u,x,y,t)$ is a given real-valued function on Ω. Assume that

$$
\begin{aligned}
&0<d_*\le d(X,t)\le d^* \quad \text{for all} \quad (X,t)\in\Omega\times J,\\
&|f'(u)|+|f''(u)|\le M, \quad u\in\mathbb{R},
\end{aligned}
\tag{1.2}
$$

where d_*, d^* and M are positive constants. If conditions (1.2) are satisfied, the problem (1.1) is uniquely solvable in a Sobolev space — cf. [42].

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In recent years, the green concept is deeply rooted in the hearts of people, and more and more new energy and environmental science issues have been studied. The geothermal resources plays a particularly important role in renewable energy research [2, 20, 24] and nonlinear parabolic equations can be used to describe the basic phenomena arising in these processes.

The finite volume element (FVE) method, also known as the generalised difference method, differs from classical finite element and finite difference methods. It is a numerical tool for solving differential equations widely used in various fields since it preserves the physical conservation laws such as mass conservation, momentum conservation, and energy conservation on all computational cells. Numerous physical problems can be transformed into differential equations. However, in many practical problems, including hydrodynamics, petroleum engineering, heat and mass transfer, it is necessary not only to ensure accuracy and easy technical implementation, but to have a certain degree of physical conservation laws. The main idea of the FVE method is to transform the area integral into the boundary integral on the dual grid. Then an appropriate test function is selected to construct a variational formula and the final discrete scheme. The theory and basic framework of the finite volume element methods have been developed in the past few decades [1, 6, 7, 9, 25–27, 32, 37].

The two-grid method proposed by Xu [47,48] in 1994 is used to deal with linear (asymmetric or indefinite) and nonlinear elliptic partial differential equations. The idea is to produce a rough approximation of the solution of nonlinear problems on a coarse grid and use it as the initial value for Newton-like iterations on a fine grid. The diameters of the coarse and fine grids, denoted by H and h, respectively, must meet the requirement $h \ll H$. Meanwhile, Huang and Chen [29] proposed a multilevel iterative method that reduces computing workload and preserves all high accuracy properties, such as superconvergence, extrapolation, etc. for finite element solutions of singular problems. Later on, two-grid method was further investigated [3,19,22,23,46]. Subsequent works deal with linear and nonlinear elliptic problems [3,10,45,51], nonlinear parabolic equations [14–16,18,41,50], fractional partial differential equations [13,31,33,35,36], nonlinear Sobolev equations [11,12,49], parabolic integro-differential equations [17, 44], and many other models [5, 28, 30, 34, 38, 40, 43].

Considering the problem (1.1), Chen *et al.* [16] studied the convergence of a first-order backward Euler scheme based on two-grid finite volume element method. Using these ideas, combined with a Crank-Nicolson scheme and two-grid method, we develop two-grid finite volume element discretisations for the nonlinear problem (1.1). The method has a high accuracy and reduces computational cost. The barycenter dual subdivision is employed to select an appropriate coarse and fine mesh matching. The nonlinear problem is solved by an iterative method on the coarse grid, while the related linear problem is solved on the fine grid with the coarse grid solution used as the initial value. We prove the convergence of the algorithm and run several examples to verify the theoretical findings. It is worth noting that the coarse mesh can be quite coarse and still maintain a good accuracy approximation. To the best of the author's knowledge, this is the first Crank-Nicolson scheme arising from two-grid finite volume element method for nonlinear parabolic equations.

The rest of this paper is organised as follows. In Section 2, we describe FVE and two-