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Conditional Regularity of Weak Solutions to the 3D Magnetic Bénard Fluid System

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Abstract. This paper concerns about the regularity conditions of weak solutions to the magnetic Bénard fluid system in $\mathbb{R}^3$. We show that a weak solution $(u,b,\theta)(\cdot,t)$ of the 3D magnetic Bénard fluid system defined in $[0,T)$, which satisfies some regularity requirement as (u,b,θ), is regular in $\mathbb{R}^3\times(0,T)$ and can be extended as a C^∞ solution beyond T.

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1 Introduction

Consider the unforced magnetic Bénard fluid system for incompressible flows on all space $\mathbb{R}^3$:

$$\begin{cases} \partial_t u+(u\cdot\nabla)u+\nabla\Big(p+\dfrac{1}{2}|b|^2\Big)=\mu\Delta u+(b\cdot\nabla)b+\theta e_3, & x\in\mathbb{R}^3,\ t\geq 0,\\ \partial_t b+(u\cdot\nabla)b=\nu\Delta b+(b\cdot\nabla)u, & x\in\mathbb{R}^3,\ t\geq 0,\\ \partial_t\theta+(u\cdot\nabla)\theta=\kappa\Delta\theta+u\cdot e_3, & x\in\mathbb{R}^3,\ t\geq 0,\\ \nabla\cdot u=\nabla\cdot b=0, & x\in\mathbb{R}^3,\ t\geq 0,\\ u(x,0)=u_0(x),\ b(x,0)=b_0(x),\ \theta(x,0)=\theta_0(x), & x\in\mathbb{R}^3. \end{cases} \tag{1.1}$$

Here $u(x,t)=(u_1(x,t),u_2(x,t),u_3(x,t))\in\mathbb{R}^3$ denotes the incompressible velocity field, $b(x,t)=(b_1(x,t),b_2(x,t),b_3(x,t))\in\mathbb{R}^3$ the magnetic field, $\theta(x,t)\in\mathbb{R}$ the temperature, $p(x,t)\in$

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$\mathbb{R}$ the hydrostatic pressure and $e_3=(0,0,1)$ the vertical unit vector. The positive constants μ, ν and κ are associated with specific properties of the fluid: The constant μ is the kinematic viscosity, ν^{-1} is the magnetic Reynolds number and κ is the coefficient of thermal diffusivity. The forcing term θe_3 in the momentum equation describes the acting of the buoyancy force on fluid motion and $u\cdot e_3$ models the Rayleigh-Bénard convection in a heated inviscid fluid. The initial data for the velocity and magnetic fields and temperature, is given by u_0, b_0 and θ_0 in (1.1), and satisfies $\nabla\cdot u_0=\nabla\cdot b_0=0$. Finally, (e_1,e_2,e_3) represents the canonical basis of $\mathbb{R}^3$.

The equations of thermohydraulics is a couple system a equations of fluid velocity and temperature. Widely known as the Bénard problem, such a system has been under intensive investigation for many decades, in particular concerning its stability (see [1] and Chapter III, Section 3.5 [2]). Furthermore, due to the necessity to consider the behavior of the thermal instability under the influence of the magnetic field, the magnetic Bénard problem has also caught much attention (e.g. [3, 4] and also [5] in the stochastic case). The system of equations (1.1) at $b\equiv\theta\equiv 0$ recovers the Navier-Stokes equations and the system of (1.1) at $\theta\equiv 0$ recovers the magnetohydrodynamics (MHD) system and the system (1.1) at $b\equiv 0$ recovers the Bénard system. All of those systems have been studied intensively in particular concerning whether given initial data sufficiently smooth, the solution remains smooth or experiences a finite time shock. We recall here, without any claim of completeness, see [7–14] and the references cited therein.

When all three parameters μ, ν and κ are positive, the global regularity of 2D magnetic Bénard fluid system follows from a standard process. However, it remains a remarkable open problem whether classical solutions of the 2D inviscid magnetic Bénard fluid system, all three parameters are zero, preserve their regularity for all time or finite time blow-up. When $\mu>0$, $\nu>0$ and $\kappa=0$, the global well-posedness result was proved by Zhou-Nakamura [15]. Cheng and Du dealt with the Cauchy problem of the 2D magnetic Bénard fluid system with mixed partial viscosity in [16]. They proved the global well-posedness of 2D magnetic Bénard fluid system without thermal diffusivity and with vertical or horizontal magnetic diffusion. Furthermore, they obtained the global regularity and some condition regularity of strong solutions for 2D magnetic Bénard fluid system with mixed partial viscosity. Particularly, the authors in [11] considered the global weak solution of the 2D Bénard system with partial dissipation and established some regularity criteria for the corresponding Bénard system.

It is currently unknown whether the solutions of 3D magnetic Bénard fluid system is globally regular (in time). The author in [17] dealt with the Cauchy problem to the 3D system of incompressible magnetic Bénard fluids, they proved that as the initial data satisfy $\|u_0\|^2_{H^1(\mathbb{R}^3)}+\|b_0\|^2_{H^1(\mathbb{R}^3)}+\|\theta_0\|^2_{H^1(\mathbb{R}^3)}\le\varepsilon$, where ε is a suitably small positive number, the three-dimensional magnetic Bénard system with mixed partial dissipation, magnetic diffusion and thermal diffusivity admit global smooth solutions. Zhang and Tang [18] established the global regularity for a special family of axisymmetric solutions to 3D magnetic Bénard fluid system.

The 2D flow generates a large family of 3D flow with vorticity stretching [19]; we