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Existence of Positive Solutions for the Nonhomogeneous Schrödinger-Poisson System with Strong Singularity

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Abstract. In this paper, a class of nonhomogeneous Schrödinger-Poisson systems with strong singularity are considered. Combining with the variational method and Nehari method, we obtain a positive solution for this system which improves the recent results in the literature.

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Key Words: Schrödinger-Poisson system; strong singularity; positive solution; Nehari method.

1 Introduction and main result

Consider the following nonhomogeneous Shrödinger-Poisson system involving singularity

$$\begin{cases} -\Delta u+\phi u=f(x)u^{-\gamma}+h(x), & x\in\Omega,\\ -\Delta\phi=u^2, & x\in\Omega,\\ u>0, & x\in\Omega,\\ u=\phi=0, & x\in\partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$, $\gamma\in(1,\infty)$ and $f\in L^1(\Omega)$ is a positive function, $h\in L^{\frac{6}{5}}(\Omega)$ is nonzero and nonnegative.

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Very recently, Yu and Chen considered system (1.1) with $h\equiv 0$, and obtained a unique positive solutions, see [1]. The singular Shrödinger-Poisson system was first studied by Zhang [2]. When $0<\gamma<1$, by using the variational method, she obtained a unique positive solution for system (1.1) with $f(x)=\mu>0, h(x)\equiv 0$. Recently, Lei and Liao generalized Theorem 1.1 in [2] to the critical case of system (1.1), and obtained two positive solutions by the variational method and Nehari manifold method, see [3].

The idea of system (1.1) is inspired by the following singular semilinear elliptic equation

$$\begin{cases} -\Delta u = g(x)u^{\beta} + f(x)u^{-\gamma}, & x\in\Omega, \\ u>0, & x\in\Omega, \\ u=0, & x\in\partial\Omega, \end{cases} \tag{1.2}$$

where Ω is a bounded domain in $\mathbb{R}^N (N\geq 3)$ with smooth boundary $\partial\Omega$, $\beta>0$, f and g are positive functions with some certain conditions. Problem (1.2) arises in the study of non-Newtonian fluids (in particular pseudoplastic fluids), boundary-layer phenomena for viscous fluids (see [4, 5]), in the Langmuir-Hinshelwood model of chemical heterogeneous catalyst kinetics (see [6]), in enzymatic kinetics models (see [7]), as well as in the theory of heat conduction in electrically conducting materials (see [8]) and in the study of guided modes of an electromagnetic field in nonlinear medium (see [9]). When $0<\gamma<1$, many papers have extensively studied problem (1.2), see for examples [10-21]. Particularly, Sun, Wu and Long studied the multiplicity positive solutions for the singular elliptic problem for the first time in [20]. Very recently, Wang and Yan considered the case of $\gamma=1$, by using the approximation method, they obtained a unique result, see [22].

While the case of problem (1.2) with $g(x)\equiv 0$ and $\gamma>1$ is first considered by Lazer and Mckenna [23], they obtained that problem (1.2) has a unique solution $u_{-\gamma}\in C^{2+\alpha}(\Omega)\cap C(\overline{\Omega})$ for any $\gamma>0$ and $f\in C^{\alpha}(\overline{\Omega})$ with $f(x)>0 (x\in\overline{\Omega})$, and $u_{-\gamma}\in H_0^1(\Omega)$ if and only if $\gamma<3$. In this direction, Boccardo and Orsina [24] studied problem (1.2) with $g(x)\equiv 0$ and $f\in L^1(\Omega)$ is nonnegative, and obtained a solution $u\in H^1_{loc}(\Omega)$ and $u^{\frac{1+\gamma}{2}}\in H_0^1(\Omega)$ for any $\gamma>1$. More recent development, when $g\in L^{\infty}(\Omega)$ and $0<q<1<\gamma$, Sun [25] obtained that problem (1.2) has a solution $u_{-\gamma}\in H_0^1(\Omega)$ provided that $f\in L^1(\Omega)$ is positive satisfying a compatible (that is (1.3)). Later, Sun and Zhang proved the relation between the standard Sobolev space $H_0^1(\Omega)$ and the critical position -3, see [26]. Particularly, very recently, Zhang [27] generalized [25] to Kirchhoff-Schrödinger-Poisson system. We also refer to [13, 14, 28-32] for $\gamma>1$.

To our best knowledge, the strong singular Schrödinger-Poisson system has not been studied up to now, except for [1] and [27]. Inspired by [2, 23, 24], we consider system (1.1) with $\gamma>1$, that is, the strong singular Schrödinger-Possion system. The main result can be described as follows.

Theorem 1.1. *Let $\Omega\subset\mathbb{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$. Assume that $\gamma\in(1,+\infty)$ and $f\in L^1(\Omega)$ is positive, and $h\in L^{\frac{6}{5}}(\Omega)$ is nonzero and nonnegative. Then, system*