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High Order Discretely Well-Balanced Methods for Arbitrary Hydrostatic Atmospheres

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Abstract. We introduce novel high order well-balanced finite volume methods for the full compressible Euler system with gravity source term. They require no à priori knowledge of the hydrostatic solution which is to be well-balanced and are not restricted to certain classes of hydrostatic solutions. In one spatial dimension we construct a method that exactly balances a high order discretization of any hydrostatic state. The method is extended to two spatial dimensions using a local high order approximation of a hydrostatic state in each cell. The proposed simple, flexible, and robust methods are not restricted to a specific equation of state. Numerical tests verify that the proposed method improves the capability to accurately resolve small perturbations on hydrostatic states.

AMS subject classifications: 76M12

Key words: Finite-volume methods, well-balancing, hyperbolic balance laws, compressible Euler equations with gravity.

1 Introduction

In many applications, the compressible Euler equations arise as a model for flow of inviscid compressible fluids such as air. Finite volume methods are commonly utilized to

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numerically approximate solutions of this system since they are conservative and capable of resolving shocks by construction. Fluid dynamics in atmospheres can be modeled by adding a gravity source term to the Euler system. This model admits non-trivial static, i.e., time independent solutions, the *hydrostatic solutions*. They are described by the *hydrostatic equation*

$$\mathbf{v}=0, \qquad \nabla p=\rho\mathbf{g}, \tag{1.1}$$

which models the balance between the gravity $\rho\mathbf{g}$, where ρ is the gas *density* and $\mathbf{g}$ is the *gravitational acceleration*, and the pressure gradient ∇p, where p is the gas *pressure*. Additionally, the solution must satisfy the constitutive relation between pressure, density, and internal energy density ε. This relation is called *equation of state* (EoS) and it has to be added to the Euler system to close it. In many practical simulations, the dynamics are considered which are close to a hydrostatic state. Standard finite volume methods usually introduce truncation errors to hydrostatic states, which can be larger than the actual perturbations related to the simulated dynamical process. Hence, the small-scale dynamics can only be resolved on very fine grids, which leads to high computational cost. This creates the demand for so-called *well-balanced* methods which are constructed to be free of a truncation error at hydrostatic states.

The idea of well-balanced methods is very common especially for the shallow water equations with bottom topography. The hydrostatic solution for the shallow water equations, the so-called lake-at-rest solution, can be given in the form of an algebraic relation. This favors the construction of well-balanced methods since the algebraic relation can be used to perform a local hydrostatic reconstruction, which is the main tool to construct well-balanced methods. Examples can be found in [1–7] and references therein. Also, for the Ripa model, which is closely related to shallow water model, there are well-balanced methods (e.g. [8,9] and references therein).

For the compressible Euler equations with gravity source term, the situation is more complicated, since the hydrostatic states are not given by an algebraic relation but by a differential equation (Eq. (1.1)) together with an EoS. Especially complicated EoS can increase the difficulty of performing a local hydrostatic reconstruction. The result is that there do not exist methods which are well-balanced for all EoS and all types of hydrostatic solutions, and all existing methods so far known for Euler equations with gravity bear some restriction. We can classify well-balanced methods broadly into three types, which may help in understanding their differences and limitations and their domain of usefulness.

In the first type of well-balancing approaches, à priori knowledge of the hydrostatic state which is to be well-balanced is assumed.

Definition 1.1 (Type 1)**.** A numerical method is well-balanced (type 1) if it exactly preserves any hydrostatic state that is given as analytical formula or in terms of discrete data on the grid.

This allows the methods to be general, such that they can balance arbitrary hydrostatic states to arbitrary EoS [10–13]. High order methods of this type are given in [14]