Ann. Appl. Math.
doi: 10.4208/aam.OA-2021-0006

Vol. **37**, No. 3, pp. 394-404
August 2021

On Fractional Smoothness of Modulus of Functions

Dong Li*

SUSTech International Center for Mathematics and Department of Mathematics, Southern University of Science and Technology, Shenzhen, Guangdong 518055, China

Received 15 July 2021; Accepted (in revised version) 8 August 2021

Abstract. We consider the Nemytskii operators $u \to |u|$ and $u \to u^{\pm}$ in a bounded domain Ω with C^2 boundary. We give elementary proofs of the boundedness in $H^s(\Omega)$ with $0 \le s < 3/2$.

AMS subject classifications: 35Q35
Key words: Nemytskii, nonlocal extension, fractional Laplacian.

1 Introduction

Let Ω be a nonempty open bounded set in $\mathbb{R}^d$. For $0<\gamma<1$ and $f \in C^1(\Omega)$, define the nonlocal $\dot{H}^\gamma$ semi-norm as

$$\|f\|^2_{\dot{H}^\gamma(\Omega)} = \int_{x,y\in\Omega, x\neq y} \frac{|f(x)-f(y)|^2}{|x-y|^{d+2\gamma}} dxdy. \tag{1.1}$$

For $f \in C^2(\Omega)$ and $0<\gamma<1$, we define

$$\|f\|_{H^{1+\gamma}(\Omega)} = \|f\|_{H^1(\Omega)} + \|\partial f\|_{\dot{H}^\gamma(\Omega)}, \tag{1.2}$$

where (and throughout this note) $\partial f = (\partial_{x_1} f, \cdots, \partial_{x_n} f)$ denotes the usual gradient. Throughout this note we shall only be concerned with real-valued functions, however with some additional work the results can be generalized to complex-valued functions. Define the Nemytskii operators

$$T_1 u = |u|, \quad T_2 u = u^+ = \max\{u,0\}, \quad T_3 u = u^- = \max\{-u,0\}. \tag{1.3}$$

The purpose of this note is to give an elementary proof of the following.

**Email:* `lid@sustech.edu.cn` (D. Li)

Theorem 1.1 (Boundedness in $H^{\frac{3}{2}-}(\Omega)$). *Let $d\ge 1$ and $0\le s<\frac{3}{2}$. Assume Ω is a nonempty open bounded set in $\mathbb{R}^d$ with C^2 boundary, i.e., locally it can be written as the graph of a C^2 function on $\mathbb{R}^{d-1}$. Then T_i, $i=1,2,3$ are bounded on $H^s(\Omega)$. More precisely,*

$$\sum_{i=1}^{3}\|T_i u\|_{H^s(\Omega)}\le \alpha_1\|u\|_{H^s(\Omega)}, \tag{1.4}$$

where $\alpha_1>0$ depends on (s,Ω,d).

Remark 1.1. In Theorem 1.1, the case $0<s<1$ is trivial thanks to the simple inequality $||x|-|y||\le|x-y|$ for any $x,y\in\mathbb{R}$. The case $s=1$ corresponds to the well-known distributional calculation $\partial(|u|)=\mathrm{sgn}(u)\partial u$ for $u\in H^1$. Thus only the case $1<s<\frac{3}{2}$ requires some work. The obstruction $s<\frac{3}{2}$ is clear since there are jump discontinuities of the gradient along manifolds of dimension $d-1$. In 1D one can take a smooth compactly supported function ϕ such that $\phi(x)\equiv x$ for x near the origin. It is trivial to verify that $|\phi|\notin H^{\frac{3}{2}}$.

Remark 1.2. It follows from our proof that for $1<s<\frac{3}{2}$, T_i maps bounded sets in $H^s(\Omega)$ to pre-compact sets in $H^1(\Omega)$. This fact has important applications in the convergence of approximating solutions in some nonlinear PDE problems.

There is by now an enormous body of literature on extension, composition, regularity and stability of nonlocal operators and we shall not give a survey on the state of art in this short note. For C^∞ boundary $\partial\Omega$, one can use interpolation to define the fractional spaces $H^s(\Omega)$ which can be regarded as restrictions of functions in $H^s(\mathbb{R}^n)$ (cf. [4]). In [6] Bourdaud and Meyer proved the boundedness of T_1 in Besov spaces $B^s_{p,p}(\mathbb{R}^d)$, $0<s<1+\frac{1}{p}$, $1\le p\le\infty$. By using linear spline approximation theory, Oswald [8] showed that T_1 is bounded in $B^s_{p,q}(\mathbb{R})$, $1\le p,q\le\infty$ if and only if $0<s<1+\frac{1}{p}$. In [9], Savaré showed the regularity of T_2 in the space $BH(\Omega)=\{u\in W^{1,1}(\Omega): D^2u \text{ is a matrix-valued bounded measure}\}$, i.e., $\nabla u\in \mathrm{BV}(\Omega;\mathbb{R}^d)$. One should note that in general T_1 increases the H^s norm for $1<s<\frac{3}{2}$. For example, for $1<s<\frac{3}{2}$, R. Musina and A. I. Nazarov [7] showed that if $u\in H^s(\mathbb{R}^d)$ changes sign, then

$$\langle(-\Delta_{\mathbb{R}^d})^s|u|,|u|\rangle>\langle(-\Delta_{\mathbb{R}^d})^s u,u\rangle. \tag{1.5}$$

We refer to [1,7] and the references therein for a more detailed survey of composition operators in function spaces with various fractional order of smoothness.

The rest of this note is organized as follows. In Section 2 we give the proof for the one dimensional case. In Section 3 we give the details for general dimensions $d\ge 2$.