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Optimal Convergence Rate of θ–Maruyama Method for Stochastic Volterra Integro-Differential Equations with Riemann–Liouville Fractional Brownian Motion

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Abstract. This paper mainly considers the optimal convergence analysis of the θ–Maruyama method for stochastic Volterra integro-differential equations (SVIDEs) driven by Riemann–Liouville fractional Brownian motion under the global Lipschitz and linear growth conditions. Firstly, based on the contraction mapping principle, we prove the well-posedness of the analytical solutions of the SVIDEs. Secondly, we show that the θ–Maruyama method for the SVIDEs can achieve strong first-order convergence. In particular, when the θ–Maruyama method degenerates to the explicit Euler–Maruyama method, our result improves the conclusion that the convergence rate is $H+\frac{1}{2}$, $H\in(0,\frac{1}{2})$ by Yang et al., J. Comput. Appl. Math., 383 (2021), 113156. Finally, the numerical experiment verifies our theoretical results.

AMS subject classifications: 65C30, 65C20, 65L20

Key words: Stochastic Volterra integro-differential equations, Riemann–Liouville fractional Brownian motion, well-posedness, strong convergence.

1 Introduction

Volterra integro–differential equations play an important role in biology, physics and engineering [1–4] and other aspects, especially in the study of heat conduction [3]. With the continuous development of science and technology [5–9], researchers have put forward many questions about Volterra integro-differential equations from practical problems. In 1966, Barnes and Allan [10] gave a simple definition of fractional Brownian motion based

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on the Riemann–Liouville integral, then the fractional Brownian motion gradually attracted much attention. The fractional Brownian motion of the Riemann–Liouville type was written by

$$B_H(t)=\frac{1}{\Gamma(H+\frac{1}{2})}\int_0^t (t-s)^{H-\frac{1}{2}}dB(s), \quad t\geq 0,$$

where $\Gamma(\cdot)$ is a Gamma function, $H\in(0,1)$, B(s) is an m-dimensional standard Wiener process defined on the complete probability space $(\Omega,\mathcal{F}_t,\{\mathcal{F}_t\}_{t\geq 0},P)$. When $H=1/2$, $B_H(t)$ degenerates into the standard Brownian motion; When $H\in(0,\frac{1}{2})$, $B_H(t)$ is not a semimartingale, and the increment is relevant due to singularity [11–13]. These properties of fractional Brownian motion bring about widespread attention, and fractional Brownian motion is used in physics, statistics, engineering, options [14–16] in the following decades. In fact, differential equations driven by fractional Brownian motion have become important mathematical models including Cox-Ingersoll-Ross model, etc. [12, 17–21]. Therefore, Volterra integro-differential equations with fractional Brownian motion have great research significance.

This paper mainly considers the nonlinear singular stochastic Volterra integro-differential equations (SVIDEs)

$$\begin{cases} \dfrac{dx(t)}{dt}=f(x(t))+\int_0^t(t-\tau)^{H-\frac{1}{2}}g(x(\tau))dB(\tau), \quad t\in[0,T], \\ x(0)=x_0, \end{cases} \tag{1.1}$$

where $f:\mathbb{R}^d\to\mathbb{R}^d$, $g:\mathbb{R}^d\to\mathbb{R}^{d\times m}$ are Borel measurable real-valued functions, $H\in(0,\frac{1}{2})$. Yang et al. [19] firstly considered the linear case of SVIDEs (1.1) and gave the strong convergence order of the Euler–Maruyama (EM) method, which is $\min\{H+\frac{1}{2},1\}$ $(0<H<1)$. Based on [19], the purpose of this paper is as follows:

- Because the well-posedness of SVIDEs (1.1) was left over from literature [19], this paper firstly proves that (1.1) has a unique strong solution. The tool used in the proof is the contraction mapping principle [22–25].
- We investigate the strong convergence order of the θ–Maruyama method, which improves the corresponding result in [19].

In fact, some progresses have been made in the strong convergence order of numerical methods for other classes of SVIDEs [26–29].

As shown in Section 2, (1.1) can be rewritten as the stochastic Volterra integral equations (SVIEs)

$$x(t)=x(0)+\int_0^t f(x(s))ds+\int_0^t \frac{1}{H+\frac{1}{2}}(t-s)^{H+\frac{1}{2}}g(x(s))dB(s), \tag{1.2}$$

where $t\in[0,T]$. It is worth emphasizing that the kernel function of (1.2) is not Lipschitz continuous, but Hölder continuous with index $H+\frac{1}{2}$, $H\in(0,\frac{1}{2})$. Indeed, for the strong