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Generalized T-Product Tensor Bernstein Bounds

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Abstract. Since Kilmer et al. introduced the new multiplication method between two third-order tensors around 2008 and third-order tensors with such multiplication structure are also called as T-product tensors, T-product tensors have been applied to many fields in science and engineering, such as low-rank tensor approximation, signal processing, image feature extraction, machine learning, computer vision, and the multi-view clustering problem, etc. However, there are very few works dedicated to exploring the behavior of random T-product tensors. This work considers the problem about the tail behavior of the unitarily invariant norm for the summation of random symmetric T-product tensors. Majorization and antisymmetric Kronecker product tools are main techniques utilized to establish inequalities for unitarily norms of multivariate T-product tensors. The Laplace transform method is integrated with these inequalities for unitarily norms of multivariate T-product tensors to provide us with Bernstein bound estimation of Ky Fan k-norm for functions of the symmetric random T-product tensors summation. Finally, we also apply T-product Bernstein inequality to bound Ky Fan norm of covariance T-product tensor induced by hypergraph signal processing.

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Key words: T-product tensors, T-eigenvalues, T-singular values, Bernstein bound, Courant-Fischer theorem for T-product tensors.

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1 Introduction

Since Kilmer et al. introduced the new multiplication method between two third-order tensors (T-product tensors), many new algebraic properties about such new multiplication rule between two third-order tensors are investigated [9, 11]. These useful algebraic properties of T-product tensors have been discovered as powerful tools in many science and engineering fields [21, 28]. Although T-product tensors have attracted many practical applications, all of these applications of T-product tensors assume that T-product tensors under consideration are deterministic. This assumption is not practical in general scientific and engineering applications based on T-product tensors. In [4, 5], the authors have tried to establish several new tail bounds for sums of random T-product tensors. These probability bounds characterize large-deviation behavior of the extreme T-eigenvalue of the sums of random T-product tensors (definitions about T-eigenvalues and T-singular values associated to T-product tensors are given in Section 2.1). The authors first apply Lapalace transform method and Lieb's concavity theorem for T-product tensors obtained from the work [4] to build several inequalities based on random T-product tensors, then utilize these inequalities to generalize the classical bounds associated with the names Chernoff, and Bernstein from the scalar to the T-product tensor setting. Tail bounds for the norm of a sum of random rectangular T-product tensors are also derived from corollaries of random symmetric T-product tensors cases. The proof mechanism is also applied to T-product tensor valued martingales and T-product tensor-based Azuma, Hoeffding and McDiarmid inequalities are also derived [5]. The random tensor and its applications in MRI and the tensor normal distribution can be found in [1, 23].

In this work, we will apply majorization techniques to establish new Bernstein bounds based on the summation of random symmetric T-product tensors. Compared to the previous work studied in [4, 5, 14], we make following generalizations: (1) besides bounds related to extreme values of T-eigenvalues, we consider more general unitarily invariant norm for T-product tensors; (2) the bounds derived in [5] can only be applied to the identity map for the summation of random symmetric T-product tensors, this work can derive new bounds for any polynomial function raised by any power greater or equal than one for the summation of random symmetric T-product tensors. In order to drive these new bounds, we also establish Courant-Fischer min-max theorem for T-product tensors in Theorem 2.1 and marjoization relation for T-singular values in Lemma 4.1. Our main theorem is provided below:

Theorem 1.1 (Generalized T-product tensor Bernstein bound)**.** *Consider a sequence* $\{\mathcal{X}_j \in \mathbb{R}^{m\times m\times p}\}$ *of independent, random symmetric T-product tensors with random structure defined by Definition* 4.1*. Let* g *be a polynomial function with*