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# Gradient Bounds for Almost Complex Special Lagrangian Equation with Supercritical Phase

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**Abstract.** In this article, we will consider the Dirichlet problem for special Lagrangian equation on $\Omega \subset M$, where $(M,J)$ is a compact almost complex manifold. Under the existence of $C^2$-smooth strictly $J$–plurisubharmonic subsolution $\underline{u}$, in the supercritical phase case, we obtain a uniform global gradient estimate.

**AMS subject classifications**: 53C07, 32Q60, 35B50.

**Key words**: Special Lagrangian equation, almost complex manifold, gradient estimates, maximum principle.

## 1 Introduction

In this paper, we consider the Dirichlet problem for the special Lagrangian operator, in the real setting which can be written as the the form

$$\sum_i \arctan \lambda_i(D^2u) = \Theta. \tag{1.1}$$

Here $\Theta$ is a topological constant called the phase angle, $D^2u$ is the real Hessian of $u$. Under the coordinate system $\{x_1,\cdots,x_n\}$, $D^2u$ can be regarded as the matrix $\{\frac{\partial^2 u}{\partial x_i \partial x_j}\}$. After an elementary orthogonal transformation,

$$\left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right) = \mathrm{diag}(\lambda_1,\cdots,\lambda_n), \quad \lambda_i := \lambda_i(D^2u).$$

If $\Theta \in \big((n-2)\frac{\pi}{2}, n\frac{\pi}{2}\big)$ (respectively $\Theta \in \big((n-1)\frac{\pi}{2}, n\frac{\pi}{2}\big)$), then we called Equation (1.1) as the special Lagrangian equation with supercritical (respectively hypercritical) phase.

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Eq. (1.1) was introduced by Harvey–Lawson [10] when they studied the Calibrated geometry. In this case, the gradient graph $x \mapsto (x, Du(x))$ defines a calibration and also determines a Lagrangian graph, which is a minimal submanifold of $\mathbb{R}^n \times \mathbb{R}^n$. In fact, any $C^2$ Lagrangian submanifold $M^{2n}$ is locally represented by a gradient graph $(x, Du(x))$ over its tangent plane. The interested reader can also refer to the Warren's PhD thesis [25] for more details of this topic. The investigation of Lagrangian manifolds also has many important applications in both geometry and physics, especially the works done by Strominger *et al.* [24] about mirror symmetry , which gave a very geometric picture of how mirror manifolds are connected.

In the real setting, the Dirichlet problem for Eq. (1.1) on smooth domain with strictly pseudoconvex boundary was also considered by Cafferalli–Nirenberg–Spruck [1] when $\Theta \in ((n-1)\frac{\pi}{2}, n\frac{\pi}{2}))$. Moreover, they also showed that the special Lagrangian operator is concave in this setting. After that, the special Lagrangian equation with supercritical phase has also been studied extensively in the past few years. For instance, Warren-Yuan [26] considered the interior gradient estimates, and the interior second order estimates were obtained by Wang–Yuan [27]. For special Lagrangian equations with more general phases, one can refer a serious works of Harvey-Lawson [11, 12, 14] *et al.* and references therein.

At the same time, in the complex setting, there were also many excellent works. For instance, Collins–Picard–Wu [4] obtained the existence and regularity theorems under the existence of subsolution for the Dirichlet problem, in both of real and complex cases. Dinew–Do–Tô [8] also obtained a continuous viscosity solution by using the classical Perron's envelope method. The complex special Lagrangian equation has an intimate connection with the deformed Hermitian-Yang-Mills equation. The interested reader can refer to [17–19] *et al.* for more profound understanding.

It is remarkable that the almost complex manifold has been studied extensively during past few years, which is motivated by differential geometry and mathematical physics ([6, 13] and references therein). In the current note, we wish to investigate the Dirichlet problem for special Lagrangian equation on the almost complex manifold.

Let $(M, J)$ be a compact almost complex manifold of real dimension $2n$, and $\Omega \subset M$ be a smooth domain with smooth boundary $\partial\Omega$. Fix a Hermitian metric $\omega$ on $M$. We wish to consider the Dirichlet problem for the almost complex special Lagrangian operator which can be written in the following form

$$\begin{cases} \sum\limits_i \arctan \lambda_i(\partial\overline{\partial}u) = h & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Here $\varphi, h$ are given functions on $\overline{\Omega}$, $\lambda_1, \cdots, \lambda_n$ are the eigenvalues of $\sqrt{-1}\partial\overline{\partial}u$ with respect to $\omega$.

We now state our main result. Assume $n\frac{\pi}{2} > h > (n-2)\frac{\pi}{2}$, i.e., the *supercritical* phase case, we have the following global gradient estimates, under the existence of $C^2$ subsolution.