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A Discontinuous Galerkin Finite Element Method without Interior Penalty Terms

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Abstract. A conforming discontinuous Galerkin finite element method was introduced by Ye and Zhang, on simplicial meshes and on polytopal meshes, which has the flexibility of using discontinuous approximation and an ultra simple formulation. The main goal of this paper is to improve the above discontinuous Galerkin finite element method so that it can handle nonhomogeneous Dirichlet boundary conditions effectively. In addition, the method has been generalized in terms of approximation of the weak gradient. Error estimates of optimal order are established for the corresponding discontinuous finite element approximation in both a discrete H^1 norm and the L^2 norm. Numerical results are presented to confirm the theory.

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1 Introduction

We will introduce a discontinuous Galerkin finite element method without a stabilizing/penalty term in this paper. A Poisson equation is considered to demonstrate the idea. However this penalty free DG method can also be used for solving other partial differential equations. We seek an unknown function u satisfying

$$-\Delta u = f \qquad \text{in } \Omega, \tag{1.1a}$$
$$u = g \qquad \text{on } \partial\Omega, \tag{1.1b}$$

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where Ω is a bounded polytopal domain in $\mathbb{R}^d$, $d=2,3$. The weak form of the problem (1.1a)-(1.1b) is given as follows: find $u\in H^1(\Omega)$ such that $u=g$ on $\partial\Omega$ and

$$(\nabla u,\nabla v)=(f,v),\quad \forall v\in H_0^1(\Omega). \tag{1.2}$$

In the past two decades, the most active research area in finite element community is the development of finite element methods with discontinuous approximation. Researchers started to use discontinuous approximation in finite element procedures in the early 1970s [2,3,8,16,20]. Local discontinuous Galerkin methods were introduced in [5]. Then a paper [1] in 2002 provided a unified analysis of discontinuous Galerkin finite element methods for the Poisson equation. Since then, many new finite element methods with discontinuous approximation have been developed, such as the hybridizable discontinuous Galerkin method [4], the mimetic finite difference method [10], the hybrid high-order method [7], the weak Galerkin method [17] and references therein. These methods in common are using discontinuous piecewise polynomials to approximate PDE's solutions. Elimination of continuity across element boundary in the finite element approximation makes it easier to work on general polygonal/polyhedral meshes in the finite element discretization.

However, one obvious drawback of discontinuous Galerkin finite element methods is their rather complicated formulations. The complexity of the DG formulation is partially due to one cannot take strong derivative on totally discontinuous polynomials. Therefore well defined weak derivatives for discontinuous polynomials were introduced in the weak Galerkin methods [17,18]. With appropriately defined weak derivatives, a discontinuous Galerkin finite element method with an ultra simple formulation has been developed in [22] for simplicial meshes and in [23] for general polytopal meshes:

$$(\nabla_w u_h,\nabla_w v)=(f,v),\quad \forall v\in V_h. \tag{1.3}$$

A penalty free DG method is studied in [9] for second order elliptic problems on simplicial meshes. Another attempt to have a simple formulation like (1.3) is the gradient discretization methods introduced in [6] by including the stabilizing terms in the definition of gradient discretization.

The finite element methods in [22,23] have the flexibility of using discontinuous approximation and, at the same time, have a simple, parameter independent finite element formulation. However these discontinuous Galerkin finite element methods have some difficulties to handle nonhomogeneous boundary conditions. In [23], all the element $T\in\mathcal{T}_h$ is assumed with no more than two edges on $\partial\Omega$ in 2D, or no more than 3 faces on $\partial\Omega$ in 3D. One of the purposes of this paper is to lift such restrictions and to handle nonhomogeneous boundary condition effectively. Another is generalizing the definitions of the weak gradient in [22,23].