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# Stochastic Runge-Kutta–Munthe-Kaas Methods in the Modelling of Perturbed Rigid Bodies

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**Abstract.** In this paper we present how nonlinear stochastic Itô differential equations arising in the modelling of perturbed rigid bodies can be solved numerically in such a way that the solution evolves on the correct manifold. To this end, we formulate an approach based on Runge-Kutta–Munthe-Kaas (RKMK) schemes for ordinary differential equations on manifolds.

Moreover, we provide a proof of the mean-square convergence of this stochastic version of the RKMK schemes applied to the rigid body problem and illustrate the effectiveness of our proposed schemes by demonstrating the structure preservation of the stochastic RKMK schemes in contrast to the stochastic Runge-Kutta methods.

**AMS subject classifications**: 60H10, 70G65, 91G80

**Key words**: Stochastic Runge-Kutta method, Runge-Kutta–Munthe-Kaas scheme, nonlinear Itô SDEs, rigid body problem

## 1 Introduction

We consider the nonlinear Itô stochastic differential equation (SDE) of the form

$$dy_t = \left( F_0(y_t) + \frac{1}{2}\sum_{i=1}^{m} F_i'(y_t)F_i(y_t) \right) dt + \sum_{i=1}^{m} F_i(y_t)\,dW_t^i, \quad y_0 \in \mathcal{M}, \tag{1.1}$$

where the solution $y_t$, $t\geq 0$, evolves on an $n$-dimensional, homogeneous submanifold $\mathcal{M}$ of $\mathbb{R}^N$, $F_i\colon \mathcal{M}\to T\mathcal{M}$ for $i=0,\cdots,m$ and $W_t^1,\cdots,W_t^m$ are independent Wiener processes. A solution can be locally defined via $y_t=\Lambda(\exp(\Omega_t),y_0)$, where $\Lambda\colon \mathcal{G}\times\mathcal{M}\to\mathcal{M}$ is a Lie

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group action on $\mathcal{M}$, i.e., for two elements $y_1,y_2 \in \mathcal{M}$ we can find a matrix $G$, an element of the Lie group $\mathcal{G}$, such that $\Lambda(G,y_1)=y_2$.

The variable $\Omega_t$ is an element of the corresponding Lie algebra $\mathfrak{g}$, which is the tangent space at the identity $e$ of $\mathcal{G}$, i.e., $\mathfrak{g}=T\mathcal{G}|_e$. It satisfies

$$d\Omega_t = A_t dt + \sum_{i=1}^{m} \Gamma_t^{(i)} dW_t^i, \quad \Omega_0 = 0, \tag{1.2}$$

where the coefficients $A_t, \Gamma_t^{(i)} \in \mathfrak{g}$ depend on the coefficients of Eq. (1.1), $F_i\colon \mathcal{M} \to T\mathcal{M}$, $i=0,\cdots,m$. We refer to [6] for more details on a general representation of these coefficients and these SDEs. A specific representation for the case $\mathcal{M}=S^2$ can be found in Section 2.

Our aim is to exploit the Euclidean-like geometry of the Lie algebra by applying stochastic Runge-Kutta (sRK) schemes to Eq. (1.2) and projecting the numerical solution back onto the manifold $\mathcal{M}$ to express an approximation of the solution of the SDE Eq. (1.1) since a direct application of sRK schemes to Eq. (1.1) would result in a drift-off. This approach is based on the Runge-Kutta–Munthe-Kaas (RKMK) schemes for ordinary differential equations (ODEs) on manifolds [11]. Their application to rigid body equations has been analyzed in [2].

Stochastic extensions of RKMK methods and their proof of convergence have already been considered in [1, 6, 10, 12]. The authors of [6] focus on the convergence of the exponential Lie series, while the authors of [1] consider only weak convergence. The proof of convergence in [12] applies only to the Euler-Maruyama scheme on matrix Lie groups and the proof of strong convergence in [10] is restricted to linear SDEs on matrix Lie groups which occur for example in the approximation of correlation matrices [9].

In this paper we extend the idea of Munthe-Kaas to SDEs on homogeneous manifolds and give a proof of the mean-square convergence of stochastic Runge-Kutta–Munthe-Kaas (sRKMK) schemes for nonlinear Itô SDEs of the form Eq. (1.1) occurring in the modelling of perturbed rigid bodies. We will show that the mean-square order of convergence $\gamma$ depends on the order of convergence of the applied sRK method in the Lie algebra and the truncation index in the series representation of the drift and diffusion coefficients of Eq. (1.2).

The structure of the paper is as follows. In Section 2 we formulate based on the deterministic case the SDE that describes the motion of a rigid body that is perturbed by stochastic processes. Then, in Section 3 we present the schemes to solve this SDE numerically such that the numerical solution evolves on the correct manifold. The results of simulating the rigid body problem are provided in Section 4. At last, a conclusion of our findings and an outlook are given in Section 5.

## 2 The stochastic rigid body problem

Let $\mathcal{M}$ be the $n$-sphere $S^n = \{y \in \mathbb{R}^{n+1} : y^\top y = 1\}$. Then the Lie group action $\Lambda$, i.e., the transport across this manifold, can be described via the matrix-vector product $\Lambda(G,y)=$