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Finite Element Error Estimation for Parabolic Optimal Control Problems with Pointwise Observations

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Abstract. In this paper, we consider parabolic distributed control problems with cost functional of pointwise observation type either in space or in time. First, we show the well-posedness of the optimization problems and derive the first order optimality systems, where the adjoint state can be expressed as the linear combination of solutions to two backward parabolic equations that involve the Dirac delta distribution as source either in space or in time. Second, we use a space-time finite element method to discretize the control problems, where the state variable is approximated by piecewise constant functions in time and continuous piecewise linear polynomials in space, and the control variable is discretized by following the variational discretization concept. We obtain a priori error estimates for the control and state variables with order $\mathcal{O}(k^{\frac{1}{2}} + h)$ up to a logarithmic factor under the L^2-norm. Finally, we perform several numerical experiments to support our theoretical results.

AMS subject classifications: 49J20, 65N15, 65N30

Key words: Parabolic optimal control problem, pointwise observation, space-time finite element method, parabolic PDE with Dirac measure, error estimate.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ be a convex polygonal or polyhedron domain, and let $T > 0$ be a constant. We consider an optimal control problem of parabolic type where the cost functional involves pointwise values of the state variable either in space or in time. The model of controlled system is characterized by the parabolic equation

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$$
\begin{aligned}
&\partial_t y - \Delta y = \mathcal{B}u && \text{in } (0,T)\times\Omega,\\
&y = 0 && \text{on } (0,T)\times\partial\Omega,\\
&y(0) = 0 && \text{in } \Omega,
\end{aligned} \tag{1.1}
$$

where y and u are the state and control variables, respectively. For convenience, we denotes by $I=(0,T)$ the time interval and by $U=L^2(I\times\omega)$ the control space, where the subdomain $\omega\subseteq\Omega$ with positive measure is called the control domain. We impose additional pointwise constraints on the control variable u and define the admissible control set as follows:

$$
U_{ad} := \big\{ u\in U : u_a \le u(t,x) \le u_b, \ \text{a.e.}, \ (t,x)\in I\times\omega \big\}
$$

with $-\infty < u_a < u_b < +\infty$. The control operator $\mathcal{B}$ maps U to $L^2(I\times\Omega)$ and is a linear bounded operator. For example, if $\omega=\Omega$, then we can take $\mathcal{B}$ as the identity operator; otherwise, $\mathcal{B}$ can be defined as a zero extension operator.

The cost functional of pointwise tracking type can be defined as

$$
\mathcal{J}_\beta(y,u) := (2-\beta)J_1(y,u) + (\beta-1)J_2(y,u) + \frac{\alpha}{2}\int_0^T\int_\omega |u(t,x)|^2 dxdt. \tag{1.2}
$$

Here J_1 and J_2 denote respectively the spatial and time observations of state, defined by

$$
\begin{aligned}
J_1(y,u) &= \frac{1}{2}\sum_{i=1}^{N_1}\int_0^T \big|y(x^i,t) - y^i_{d_S}(t)\big|^2 dt,\\
J_2(y,u) &= \frac{1}{2}\sum_{i=1}^{N_2}\int_\Omega \big|y(t^i) - y^i_{d_T}\big|^2 dx + \frac{1}{2}\int_\Omega \big(y(T)-y_T\big)^2 dx,
\end{aligned}
$$

where $N_1, N_2 \in N_+$, $y_T \in L^2(\Omega)$, $\alpha > 0$ is a regularization parameter and N_+ denotes the set of positive integers. The sets $\{x^i, i=1,\ldots,N_1\}\subset \mathrm{Int}(\Omega)$ and $\{t^i, i=1,\ldots,N_2\}\subset(0,T)$ are respectively called the set of spatial observation points and the set of time observation points, and $\{y^i_{d_S}\in L^2(\Omega),\ i=1,\ldots,N_1\}$ and $\{y^i_{d_T}\in L^2(0,T),\ i=1,\ldots,N_2\}$ are respectively called the spatial observations and the time observations. The parameter $\beta\in[1,2]$ is the weight between the pointwise spatial observations and pointwise time observations. Specifically, $\beta=1$ refers to the case of pure spatial observation and $\beta=2$ refers to the case of pure time observation, while $\beta\in(1,2)$ refers to the case with both observations and weights the importance of two observations.

With the above defined cost functional, our parameter-dependent optimal control problem reads: Find $(\bar{y},\bar{u})\in X\times U_{ad}$ such that

$$
\mathcal{J}_\beta(\bar{y},\bar{u}) \le \mathcal{J}_\beta(y,u), \quad \forall (y,u)\in X\times U_{ad} \quad \text{subject to} \quad (1.1), \tag{1.3}
$$

where X is the state space given in Section 3.