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High Order Mixed Finite Elements with Mass Lumping for Elasticity on Triangular Grids

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Abstract. A family of conforming mixed finite elements with mass lumping on triangular grids are presented for linear elasticity. The stress field is approximated by symmetric $H(\mathrm{div}) - P_k$ $(k \geq 3)$ polynomial tensors enriched with higher order bubbles so as to allow mass lumping, and the displacement field is approximated by $C^{-1} - P_{k-1}$ polynomial vectors enriched with higher order terms. For both the proposed mixed elements and their mass lumping schemes, optimal error estimates are derived for the stress and displacement in $H(\mathrm{div})$ norm and L^2 norm, respectively. Numerical results confirm the theoretical analysis.

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Key words: Linear elasticity, mixed finite element, mass lumping, error estimate.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a polygonal region with boundary $\partial\Omega$. We consider the following mixed variational system of linear elasticity based on the Helligner-Reissner principle: Find $(\sigma, u) \in \Sigma \times V := H(\mathrm{div}, \Omega; \mathbb{S}) \times L^2(\Omega; \mathbb{R}^2)$, such that

$$\begin{cases} (\mathcal{A}\sigma, \tau) + (\mathrm{div}\, \tau, u) = 0, & \forall \tau \in \Sigma, \\ -(\mathrm{div}\, \sigma, v) = (f, v), & \forall v \in V. \end{cases} \tag{1.1}$$

Here $\sigma : \Omega \to \mathbb{S} := \mathbb{R}^{2\times 2}_{\mathrm{sym}}$ denotes the symmetric 2×2 stress tensor field, $u : \Omega \to \mathbb{R}^2$ the displacement field, and $\mathcal{A}\sigma \in \mathbb{S}$ the compliance tensor with

$$\mathcal{A}\sigma := \frac{1}{2\mu}\left(\sigma - \frac{\lambda}{2\mu + 2\lambda}\,\mathrm{tr}(\sigma) I\right), \tag{1.2}$$

where $\lambda > 0, \mu > 0$ are the Lamé coefficients, $\mathrm{tr}(\sigma)$ the trace of σ, I the 2×2 iden-

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tity matrix, and f the body force. $H(\mathrm{div},\Omega;\mathbb{S})$ denotes the space of square-integrable symmetric matrix fields with square-integrable divergence, and $L^2(\Omega;\mathbb{R}^2)$ the space of square-integrable vector fields. The L^2 inner products on vector and matrix fields are given by

$$(v,w) := \int_\Omega v\cdot w dx = \int_\Omega \sum_{i=1}^{2} v_i w_i dx, \qquad v=(v_1,v_2), \quad w=(w_1,w_2)\in V,$$
$$(\sigma,\tau) := \int_\Omega \sigma : \tau dx = \int_\Omega \sum_{1\le i,j\le 2} \sigma_{ij}\tau_{ij} dx, \quad \sigma=(\sigma_{ij}), \qquad \tau=(\tau_{ij})\in\Sigma,$$

respectively.

According to the standard theory of mixed methods [11], a mixed finite element discretization of the weak problem (1.1) requires the pair of stress and displacement approximations to satisfy two stability conditions, i.e. a coercivity condition and an inf-sup condition. These stability constraints make it challengeable to construct stable finite element pairs with symmetric stresses. In this field, we refer to [1–7,12,20–26,31, 32] for some conforming or nonconforming mixed methods for elasticity. In particular, Hu and Zhang [25, 26] designed a family of conforming symmetric mixed finite elements with optimal convergence orders for linear elasticity on triangular and tetrahedral grids. Later Hu [21] extended the elements to simplicial grids in $\mathbb{R}^n$ for any positive integer n. In these elements, the stress is approximated by symmetric $H(\mathrm{div},\Omega;\mathbb{S})-P_k$ polynomial tensors and the displacement is approximated by $L^2(\Omega;\mathbb{R}^n)-P_{k-1}$ polynomial vectors for $k\ge n+1$.

However, for a mixed finite element discretization based on (1.1), a computational drawback is the need to solve an algebraic system of saddle point type like

$$\begin{pmatrix} \mathbb{A} & \mathbb{B}^T \\ -\mathbb{B} & \mathbb{O} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} O \\ F \end{pmatrix}, \tag{1.3}$$

where $\mathbb{A}$ is a symmetric and positive definite (SPD) matrix corresponding to the term $(\mathcal{A}\sigma,\tau)$ in (1.1), and X_1 and X_2 are the vectors of unknowns for the discrete stress and displacement approximations, respectively. One possible approach to resolve this difficulty is to apply "mass lumping" on $(\mathcal{A}\sigma,\tau)$ so as to get a diagonal or block-diagonal matrix approximation, $\tilde{\mathbb{A}}$, of the 'mass matrix' $\mathbb{A}$. Replacing $\mathbb{A}$ with $\tilde{\mathbb{A}}$ in the discrete system (1.3), we obtain

$$X_1 = -\tilde{\mathbb{A}}^{-1}\mathbb{B}^T X_2$$

and then

$$\mathbb{B}\tilde{\mathbb{A}}^{-1}\mathbb{B}^T X_2 = F. \tag{1.4}$$

Notice that $\tilde{\mathbb{A}}$ is diagonal or block-diagonal, so is $\tilde{\mathbb{A}}^{-1}$. This means that the Schur complement $\mathbb{B}\tilde{\mathbb{A}}^{-1}\mathbb{B}^T$ is SPD. As a result, by mass lumping the saddle point system (1.3) is reduced to the SPD system (1.4), which can be solved efficiently by many fast algorithms.