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Band-Times-Circulant Preconditioners for Non-Symmetric Real Toeplitz Systems with Unknown Generating Function†

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Abstract. In this paper we study the preconditioning of $n \times n$ non-symmetric, real Toeplitz systems, when the generating function of the coefficient matrix T_n is not known a priori, but we know that a generating function f exists related to the matrix sequence $\{T_n\}$, $T_n = T_n(f)$, with f smooth enough. The proposed preconditioner is derived as a combination of a band Toeplitz and a circulant matrix. We give details for the construction of the proposed preconditioner, by the entries of T_n and we study the cluster of the eigenvalues, as well as of the singular values, of the sequences of the coefficient matrices related to the preconditioned systems. Theoretical results prove the efficiency of the Preconditioned Generalized Minimal Residual method (PGMRES) and the Preconditioned Conjugate Gradient method of Normal Equations (PCGN). Such efficiency is also shown in demonstrating numerical examples, using the proposed preconditioning technique.

AMS subject classifications: 65F08, 65F10, 15B05, 15A18

Key words: Non-symmetric, Toeplitz, band-times-circulant, preconditioner.

1. Introduction

Toeplitz systems appear in a variety of applications such as image processing and restoration, systems with queuing networks and problems coming up from the discretization of ordinary and partial differential equations, as well as integro-differential equations (see [38, 40, 51]). It is well-known that a Toeplitz matrix T_n has the same entries across its diagonals, as it is displayed in (1.1). Note that the entries of a Toeplitz matrix may be given

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as the Fourier coefficients of a function f, which is called the generating function of the coefficient matrix. In case where each column/row of T_n is a circular shift of its preceding column/row, the Toeplitz matrix is called circulant. Thus, T_n is a circulant matrix if and only if it is Toeplitz and it holds $t_{ij} = t_{kl}$, if $j - i = l - k (\bmod n)$, $i, j, k, l = 1, \ldots, n$

$$T_n = \begin{bmatrix} t_0 & t_{-1} & \cdots & t_{-(n-2)} & t_{-(n-1)} \\ t_1 & t_0 & t_{-1} & \ddots & t_{-(n-2)} \\ \vdots & t_1 & \ddots & \ddots & \vdots \\ t_{n-2} & \ddots & \ddots & t_0 & t_{-1} \\ t_{n-1} & t_{n-2} & \cdots & t_1 & t_0 \end{bmatrix}. \tag{1.1}$$

Over the years, many researchers have proposed circulant preconditioners for symmetric and positive definite Toeplitz systems [20, 26, 58, 62]. For the same class of systems, someone can also find in the literature a well-known preconditioning technique, using band Toeplitz preconditioners [21, 31, 45, 52], the main idea of which is to find a trigonometric polynomial g, having the same roots and with the same multiplicities, as the generating function f and use $T_n(g)$ as a preconditioner. Practical applications and recent advances closely related to this statement are the preconditioning methods and theoretical analyses for the following: the Sinc-Galerkin linear systems with respect to second-order elliptic partial differential equations [12], the time-dependent partial differential equations [9], the Burgers equations [8] and the linear third-order ordinary differential equations [6, 7, 14]. A technique combining band Toeplitz matrices and circulant ones, was proposed in [46]. More specifically, the proposed preconditioner was a product of $T_n(g)$ (g is the trigonometric polynomial which eliminates the roots of f) with the circulant matrix having as eigenvalues the values of f/g at the points $(2(k-1)\pi)/n$, $k = 1, 2, \ldots, n$.

Recently, circulant preconditioners have been introduced for non-symmetric Toeplitz systems [33,36,47]. At this point let us note that closely related and more recent work about fast preconditioning iterative methods for solving the discrete linear systems arising from the spatial fractional diffusion equations can be found in [10, 11]. A proposed combination of band and circulant matrices, when the generating function of the Toeplitz system is known a priori, was given in [44]. This technique constitutes a generalization of [46], for non-symmetric Toeplitz systems and the circulant matrix, of the band-times-circulant product was

$$C_n\left(\frac{f}{g}\right) = \mathscr{F}_n^H \operatorname{diag}\left(\frac{f}{g}(0), \frac{f}{g}\left(\frac{2\pi}{n}\right), \ldots, \frac{f}{g}\left(\frac{2(n-1)\pi}{n}\right)\right)\mathscr{F}_n, \tag{1.2}$$

where $\mathscr{F}_n$ denotes the Fourier matrix of dimension n. Of course, $T_n(g)$ denotes the band Toeplitz matrix generated by the trigonometric polynomial g, which eliminates the roots of f. In the case where f has no roots the authors proposed the usage of the preconditioner $C_n(f)$, instead of $T_n(g)C_n(f/g)$. We refer to some interesting papers where the combination of band Toeplitz and matrix algebra preconditioners was also proposed [22, 56].