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Multi-Phase Segmentation Using Modified Complex Cahn-Hilliard Equations

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Abstract. In this paper, we propose a novel PDE-based model for the multi-phase segmentation problem by using a complex version of Cahn-Hilliard equations. Specifically, we modify the original complex system of Cahn-Hilliard equations by adding the mean curvature term and the fitting term to the evolution of its real part, which helps to render a piecewisely constant function at the steady state. By applying the K-means method to this function, one could achieve the desired multiphase segmentation. To solve the proposed system of equations, a semi-implicit finite difference scheme is employed. Numerical experiments are presented to demonstrate the feasibility of the proposed model and compare our model with other related ones.

AMS subject classifications: 68U10, 65K10, 65N06

Key words: Image segmentation, Cahn–Hilliard equation, semi-implicit finite difference scheme.

1. Introduction

Image segmentation aims to partition an image domain into a few disjoint parts, each of which represents some meaningful object or background. It is a classical problem in image processing and has been extensively studied during the past few decades. Different mathematical approaches, including stochastic and deterministic methods, have been used to solve this problem. Among these methods, variational methods or PDE-based methods were widely utilized as they could be easily adapted to the complexities and flexibilities of real applications.

In the literature, many variational models have been proposed for solving the segmentation problem, including the well-known Mumford-Shah model [17], the classical

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snake model by Kass, Witkin, and Terzopoulos [12], the geodesic active contour model by Caselles, Kimmerl, and Sapiro [5], the Chan-Vese model [6], etc. Note that the three later models all deal with the problem of two-phase segmentation, that is, the resulting segmentation contour partitions an image domain into two parts, with one for objects and the other for background. Usually, there could exist multiple objects with distinct image intensities in a given image. Then the goal of segmentation is to separate all these objects, which leads to the multi-phase segmentation problem.

There also exist many multi-phase segmentation models in the literature. For instance, Vese and Chan [20] extended their original Chan-Vese model for the segmentation of multiple objects, i.e., the partition contains more than two regions. To this end, they considered multiple level set functions [18] and used the combination of the signs of these functions to label the resulting multiple regions. For instance, n level set functions can lead to a possibility of at most 2^n regions. Despite of the efficiency of representing regions using multiple level set functions, the numerical implementation is usually expensive due to the cost for each level set function. To overcome this issue, Chung and Vese [8] considered one implicit Lipschitz continuous function defined on the image domain and used it to separate different regions through its level lines. Lie *et al.* [14] introduced the piecewise constant level set method by assigning each of the constant values for a phase of the segmentation. In both methods, only one function is needed for multi-phase segmentation, which saves the computational effort remarkably. Later on, Jung *et al.* [11] proposed a multi-phase segmentation model that is based on the phase transition model of Modica and Mortola in material sciences. This model introduced a novel fitting term that uses the sinc-function to separate different phases.

Different from all the above approaches, Cai *et al.* [4] proposed a two-stage multi-phase segmentation model. Specifically, in the first stage, they solve a convex variant of the Mumford-Shah functional given as follows:

$$\inf_g \left\{ \frac{\lambda}{2}\int_\Omega (f-\mathcal{A}g)^2 dx + \frac{\mu}{2}\int_\Omega |\nabla g|^2 dx + \int_\Omega |\nabla g| dx \right\}, \tag{1.1}$$

where $f:\Omega\to\mathbb{R}$ is a given image defined on $\Omega\subset\mathbb{R}^2$, $\mathcal{A}$ represents a linear blurring operator or the identity operator, and $\lambda,\mu>0$ are parameters. In this functional, the original length term of boundary in the Mumford-Shah functional is replaced by the total variation of the desirable clean function g. Note that this new functional is convex, and its minimizer is unique. Then, as discussed in [4], the second stage is to segment the minimizer g into K phases ($K\geq 2$) by using thresholds or by any clustering methods like the K-means method. This model possesses several merits:

1) when compared with the Mumford-Shah, the above model can be more easily handled both analytically and numerically;

2) the phase number K of segmentation can be chosen by users without re-calculating the minimizer g.