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REDUCED APPROACH FOR STOCHASTIC OPTIMAL CONTROL PROBLEMS

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Abstract. In this paper, we develop and analyze the reduced approach for solving optimal control problems constrained by stochastic partial differential equations (SPDEs). Compared to the classical approaches based on Monte Carlo method to the solution of stochastic optimal control and optimization problems, e.g. Lagrange multiplier method, optimization methods based on sensitivity equations or adjoint equations, our strategy can take best advantage of all sorts of gradient descent algorithms used to deal with the unconstrained optimization problems but with less computational cost. Specifically, we represent the sample solutions for the constrained SPDEs or the state equations by their associated inverse-operators and plug them into the objective functional to explicitly eliminate the constrains, the constrained optimal control problems are then converted into the equivalent unconstrained ones, which implies the computational cost for solving the adjoint equations of the derived Lagrange system is avoided and faster convergent rate is expected. The stochastic Burgers' equation with additive white noise is used to illustrate the performance of our reduced approach. It no doubt has great potential application in stochastic optimization problems.

Key words. SPDEs-constrained optimization problems, Lagrange multiplier method, the reduced approach, Monte Carlo finite element method.

1. Introduction

Over the past decades, the computational community has shown a growing interest in designing fast solution methods for optimal control problems constrained by stochastic partial differential equations [21,23,24,46,53]. In this case, the Monte Carlo methods are typically used in conjunction with the associated Galerkin finite element approximation in space [10,22,28] to overcome the *curse of dimensionality*, i.e., the situation in which the volume of the sample space increases exponentially with the dimension, and obtain a reliable model. However, Monte Carlo simulation typically requires a large number of sample solutions which may lead to formidable computational cost. Therefore, effective algorithms are urgently desired to solve these large-scale SPDE-constrained optimization problems in practice.

In this work, we consider a stochastic optimal control problem with tracking type objective functional. The control goal is to determine a state variable u and a deterministic control variable f to minimize

$$\mathscr{J}(u,f) := \mathbb{E}\left[\frac{1}{2}\int_0^T \|u-U\|^2_{L^2(\mathcal{D})}\,\mathrm{d}t\right] + \frac{\beta}{2}\int_0^T \|f\|^2_{L^2(\mathcal{D})}\,\mathrm{d}t \tag{1}$$

over a convex, bounded and polygonal spatial domain $\mathcal{D} \subset \mathbb{R}^d$ $(d = 2, 3)$, where U is a given expected state and usually assumed to be deterministic, $\mathbb{E}$ denotes an expected value, which is defined as the Lebesque integral in a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$ described in section 2, β is a regularization parameter. u is the

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solution to a given SPDE, i.e., the state equation, which can be written in the abstract form

$$\mathcal{A}(u, f) = 0, \tag{2}$$

where the operator $\mathcal{A}$ denotes a SPDE equipped with appropriate boundary and initial conditions. (2) is usually used to model many physical, biological and economic systems subject to the influence of randomness. In brief, the constrained optimization problem we consider is then to find states u and controls f such that the functional given in (1) is minimized subject to (2).

the minimization in (1) is constrained by (2).

In the study of turbulence phenomena, the Burgers' equations can be viewed as a simplified version of the Navier-Stokes equations. Analysis and numerical approximation of optimal control problems constrained by the Burgers' equation are thus important to a variety of more complicated optimization problems in fluid dynamics. Control problems of the deterministic Burgers' equation have been studied by many authors [7, 25, 27, 30, 42, 48, 50–52], and stochastic control problems in [2, 12, 13, 29]. Here we focus on the case of stochastic Burgers' equation with additive white Gaussian noise.

To solve the large-scale optimization problems resulted from the Monte Carlo finite element (MC-FE) discretization, the classical approaches, e.g. Lagrange multiplier method, optimization method based sensitivity equations or adjoint equations [4, 17, 19, 40, 44] require the update of gradient over the samples, thus demanding repeated and costly sample solutions of the state and adjoint equations or sensitivity equations. In practice, they are typically not feasible for large-scale optimization problems due to the unaffordable computational cost for the resulted optimization system .

In this paper, we proposed the reduced approach to the stochastic optimal control problems. The reduced approach has been used to solve PDE-constrained optimization in inverse problems [31], but there are very rare literatures exploring the application of reduced approach to the stochastic optimization problems. For the solvability of the optimal control problems, in literature, there are two different strategies: Discretize-then-Optimize approach [32, 35, 39, 40] and Optimize-then-Discretize approach [32, 36, 37, 41, 45, 46], the former approach is to discretize the continuous problem and then accordingly derive for the optimality conditions, while the latter one refers to optimality condition on the continuous level is formulated first and then discretized. In our reduced approach, the Discretize-then-Optimize strategy will be adopted. Specifically, we first discretize the objective functional and the state equations, then we represent the sample solutions for the constrained SPDEs by their associated inverse-operators $\mathcal{G}$ and plug them into the discrete objective functional to explicitly eliminate the constrains. This elimination leads to a reduced objective functional $\hat{\mathcal{J}}(\mathcal{G}(f), f)$. The derived reduced system no longer has to solve the adjoint equations, but directly obtains the gradient direction through the *chain rule*. From the optimization point of view, the reduced approach can make full use of various gradient descent algorithms for unconstrained optimization problems and has low computational cost. Numerical experiments also show that the new technique works well. Moreover, much of our results and computations can be readily extended to to other optimization control problems. Figure 1 presents the outline of our optimization algorithm.

The remainder of this paper is organized as follows. In Section 2.1, we give a brief overview of some function spaces and notations. And the approximation of Brownian white noise via piecewise constant functions are discussed in Section