

## A General Improvement in the WENO-Z-Type Schemes

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**Abstract.** A new type of finite volume WENO schemes for hyperbolic problems was devised in [33] by introducing the order-preserving (OP) criterion. In this continuing work, we extend the OP criterion to the WENO-Z-type schemes. We firstly rewrite the formulas of the Z-type weights in a uniform form from a mapping perspective inspired by extensive numerical observations. Accordingly, we build the concept of the locally order-preserving (LOP) mapping which is an extension of the order-preserving (OP) mapping and the resultant improved WENO-Z-type schemes are denoted as LOP-GMWENO-X. There are four major advantages of the LOP-GMWENO-X schemes superior to the existing WENO-Z-type schemes. Firstly, the new schemes can amend the serious drawback of the existing WENO-Z-type schemes that most of them suffer from either producing severe spurious oscillations or failing to obtain high resolutions in long calculations of hyperbolic problems with discontinuities. Secondly, they can maintain considerably high resolutions on solving problems with high-order critical points at long output times. Thirdly, they can obtain evidently higher resolution in the region with high-frequency but smooth waves. Finally, they can significantly decrease the post-shock oscillations for simulations of some 2D problems with strong shock waves. Extensive benchmark examples are conducted to illustrate these advantages.

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**Key words:** WENO-Z-type schemes, locally order-preserving mapping, hyperbolic problems.

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### 1 Introduction

During the past thirty years, the weighted essentially non-oscillation (WENO) schemes [4,5,15–17,26,27,35,36], as well as the recently-published low-dissipation shock-capturing

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ENO-family schemes, dubbed TENO [6–13], is a major area of interest within the field of high-resolution numerical simulation for the following hyperbolic conservation laws

$$\frac{\partial \mathbf{u}}{\partial t} + \sum_{\alpha=1}^d \frac{\partial \mathbf{f}_{\alpha}(\mathbf{u})}{\partial x_{\alpha}} = 0, \quad x_{\alpha} \in \mathbb{R}, \quad t > 0, \quad (1.1)$$

where  $\mathbf{u} = (u_1, u_2, \dots, u_m) \in \mathbb{R}^m$  are the conserved variables and  $\mathbf{f}_{\alpha}: \mathbb{R}^m \rightarrow \mathbb{R}^m$ ,  $\alpha = 1, 2, \dots, d$  are the Cartesian components of flux.

By using the information of all candidate substencils of the original essentially non-oscillation (ENO) scheme [1–3, 14] through a convex combination, Liu et al. [17] proposed the  $(r+1)$ th-order WENO scheme. Later, Jiang and Shu [26] improved it by introducing a new measurement of the smoothness of a solution over a particular substencil, say, local smoothness indicator (LSI), into the  $(2r+1)$ th-order one, dubbed WENO-JS. WENO-JS is the most widely used one among the family of the WENO schemes since it was proposed as it can maintain the ENO property near discontinuities and obtain the designed convergence rate of accuracy in most smooth regions. However, it was commonly known that [28–32, 37, 44, 47] WENO-JS fails to obtain the optimal accuracy near critical points of order  $n_{\text{cp}}$ , and this was originally discovered by Henrick et al. [28]. Here,  $n_{\text{cp}}$  stands for the order of the critical point for the function  $f$  that satisfies  $f' = 0, \dots, f^{(n_{\text{cp}})} = 0, f^{(n_{\text{cp}+1})} \neq 0$ . In the same article, Henrick et al. performed the truncation error analysis and derived the necessary and sufficient conditions on the nonlinear weights of the WENO schemes to achieve the designed fifth-order convergence in smooth regions of the solution. Then, they designed a mapping function to the original weights of the WENO-JS scheme resulting in the mapped weights satisfying these conditions. The resultant scheme, denoted as WENO-M, successfully recovered the designed convergence orders even at or near the critical points. It is since the introduction of Henrick et al. [28] that various versions of mapped WENO schemes, such as WENO-PMk [30], WENO-IM( $k, A$ ) [29], WENO-PPMn [38], WENO-RM( $mn0$ ) [37], WENO-MAIMi [31], WENO-ACM [47], and etc., have been developed by devising different mapping functions under the similar principles of WENO-M. Over the past decade, there has been an increasing amount of literature on the long output time simulations of mapped WENO schemes [29–31, 33, 34, 37, 46]. A key issue of WENO-M is that its resolution decreases dramatically when solving problems with discontinuities for long output times, and this drawback was first noticed and successfully fixed by Feng et al. [30]. After that, a series of mapped WENO schemes, as reported in [29–31, 37], have been developed to address this potential loss of accuracy properly. However, in these same articles, it was illustrated that these schemes have caused another serious problem that they produced severe spurious oscillations for long output time calculations because of the lack of robustness. Taken together, it is rather difficult for the previously published mapped WENO schemes to avoid spurious oscillations while obtain high resolutions at the same time for long output time simulations. The essential reason of such phenomena has been revealed in a recently published article [33] in which the core concept of *order-preserving (OP) mapping* was innovatively proposed resulting in