Ann. of Appl. Math.
34:1(2018), 1-31

GLOBAL EXISTENCE OF WEAK SOLUTIONS TO THE THREE-DIMENSIONAL FULL COMPRESSIBLE QUANTUM EQUATIONS*

Boling Guo†
(*Institute of Applied Physics and Computational Math., Beijing 100088, PR China*)
Binqiang Xie
(*The Graduate School of China Academy of Engineering Physics, Beijing 100088, PR China*)

Abstract

We consider the quantum Navier-Stokes equations for the viscous, compressible, heat conducting fluids on the three-dimensional torus T^3. The model is based on a system which is derived by Jungel, Matthes and Milisic [15]. We made some adjustment about the relation of the viscosities of quantum terms. The viscosities and the heat conductivity coefficient are allowed to depend on the density, and may vanish on the vacuum. By several levels of approximation we prove the global-in-time existence of weak solutions for the large initial data.

Keywords global weak solution; compressible quantum Navier-Stokes equations; thermal conduction

2000 Mathematics Subject Classification 76E25; 76E17; 76W05; 35Q35

1 Introduction

In this paper, we are interested in the quantum fluid models. Such models can be used to describe superfluids [18], quantum semiconductors [7], weakly interacting Bose gases [11] and quantum trajectories of Bohmian mechanics [25]. Since the numerical solution of the Schrodinger equation or the Wigner equation is very time consuming, fluid-type quantum models seem to provide a compromise between accurate and efficient numerical simulations. Moreover, quantum fluid models are formulated in macroscopic quantities like the current density, which can be measured. A hydrodynamic form of the single-state Schrodinger was already derived by Madelung [21]. Later, the so-called quantum hydrodynamic equations were derived by Ferry and Zhou [7] from the Bloch equation for the density matrix. In [12] Gardner used the

*Manuscript received December 10, 2017
†Corresponding author. E-mail: gbl@iapcm.ac.cn

moment method to the Wigner equation leading to the full three-dimensional quantum hydrodynamic model (QHD). Jungel, Matthes and Milisic [15] obtained a new quantum hydrodynamic model using Levermore's entropy minimization principle, which can be used to derive the full three-dimensional quantum hydrodynamic model including the vorticity matrix. Recently some dissipative quantum fluid models have been derived. In [13] the authors derived viscous quantum Euler models using a moment method in Wigner-Fokker-Planck equation. In [5], under some conditions, using a Chapman-Enskog expansion in Wigner equation, the quantum Navier-Stokes equations were obtained.

In the following, we consider a full quantum viscous quantum equations as follows:

$$\partial_t \rho + \mathrm{div}(\rho \mathbf{u}) = 0, \tag{1.1}$$

$$\partial_t(\rho \mathbf{u}) + \mathrm{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla P - 2\delta^2 \mathrm{div}(\rho(\nabla \otimes \nabla)\log \rho) = \nu \mathrm{div}(\rho D(\mathbf{u})), \tag{1.2}$$

$$\begin{aligned}&\partial_t(\rho E) + \mathrm{div}(\rho E \mathbf{u}) + \mathrm{div}(P\mathbf{u}) - 2\delta^2 \mathrm{div}(\rho \mathbf{u}(\nabla \otimes \nabla)\log \rho) - \delta^2 \mathrm{div}(\rho \Delta \mathbf{u})\\ &= \mathrm{div}(q) + \nu \mathrm{div}(\rho D(\mathbf{u})\mathbf{u}),\end{aligned} \tag{1.3}$$

with the total energy, the thermal diffusion flux and symmetric part of the velocity gradient respectively,

$$\rho E = \rho e + \frac{1}{2}\rho|u|^2 - \delta^2 \rho \Delta \log \rho, \quad q = \kappa(\rho,\theta)\nabla\theta, \quad D(\mathbf{u}) = \frac{\nabla \mathbf{u} + \nabla^T \mathbf{u}}{2},$$

where ρ is the density of the fluid, $\mathbf{u}$ denotes the velocity field of the fluid, θ is the temperature of the fluid, P is the pressure field, q is the diffusion flux, κ is the thermal conductivity coefficient. The physical parameters are the Plank constant $\delta^2 > 0$ and the viscosity constant $\nu > 0$. This system of equations corresponds to Garder's QHD model [12] except for the dispersive terms $\delta^2\mathrm{div}(\rho\Delta\mathbf{u})$ and viscous terms $\nu\mathrm{div}(\rho D(\mathbf{u})\mathbf{u})$.

Interestingly, quantum terms can be cancelled in the total energy equation. In fact, by substituting the above expression for the total energy density into equation (1.3) yields

$$\partial_t(\rho e) + \mathrm{div}(\rho e \mathbf{u}) + P\mathbf{u} = \mathrm{div}(\kappa(\rho,\theta)\nabla\theta) + \nu\rho|D(\mathbf{u})|^2, \tag{1.4}$$

System (1.1)-(1.3) is considered under initial conditions:

$$\rho|_{t=0} = \rho_0, \quad \rho\mathbf{u}|_{t=0} = m_0, \quad \rho E|_{t=0} = (\rho E)_0.$$

Here the functions ρ_0 and m_0 satisfy:

$$m_0 = 0 \ \ a.e. \text{ on } \{x \in R^n : \rho_0 = 0\}. \tag{1.5}$$

There have been a large amount of work on the global existence of weak solutions to the compressible Navier-Stokes equation without quantum effect, in the constant