Ann. of Appl. Math.
34:1(2018), 58-70

THE SYMMETRY DESCRIPTION OF A CLASS OF FRACTIONAL STURM-LIOUVILLE OPERATOR*†

Shouyi Li‡, Zhaowen Zheng

(*School of Mathematical Science, Qufu Normal University, Qufu 273165, Shandong, PR China*)

Abstract

This paper studies the symmetry of a class of fractional Sturm-Liouville differential equations with right and left fractional derivatives. We give the Hermitian boundary condition description of this problem. Furthermore, the density of minimal operator is given. Then the symmetry of this problem is obtained.

Keywords fractional differential operator; differential operator; Sturm-Liouville; density; symmetric operator

2000 Mathematics Subject Classification 34B24; 34A08

1 Introduction

Recently, fractional differential equations have drawn much attention. It is caused both by the theory of fractional calculus itself and by the applications in various fields of science and engineering such as control, electrochemistry, electromagnetic, porous media, viscoelasticity, etc, for detail, see [1-5].

In this paper, we consider the following fractional Sturm-Liouville operator:

$$(ly)(x) = {}^{c}\mathcal{D}^{\alpha}_{1-}p(x)D^{\alpha}_{0+}y(x) + q(x)y(x), \tag{1.1}$$

acting on the Hilbert space $L^2[0,1]$ with the following boundary condition:

$$\begin{cases} a_{11}I^{1-\alpha}_{0+}y(0) + a_{12}D^{\alpha}_{0+}y(0) + b_{11}I^{1-\alpha}_{0+}y(1) + b_{12}D^{\alpha}_{0+}y(1) = 0, \\ a_{21}I^{1-\alpha}_{0+}y(0) + a_{22}D^{\alpha}_{0+}y(0) + b_{12}I^{1-\alpha}_{0+}y(1) + b_{22}D^{\alpha}_{0+}y(1) = 0, \end{cases} \tag{1.2}$$

where $p(x) \neq 0$, $p(x) \in C^1(0,1)$, $q(x) \in C(0,1)$, $0 < \alpha < 1$ and $a_{ij}, b_{ij} \in \mathbb{R}$ $(i,j = 1,2)$. $I^{1-\alpha}_{0+}$, D^{α}_{0+} and ${}^{c}\mathcal{D}^{\alpha}_{1-}$ are fractional integral operator, fractional derivative and Caputo fractional derivator acting on given functions respectively. Our purpose is

*Manuscript received June 29, 2017; Revised November 30, 2017

†This research was partially supported by the NSF of China (Grant.11271225).

‡Corresponding author. E-mail: shouyili61@163.com

to find sufficient conditions on the coefficients matrices A and B to guarantee a symmetric operator, where

$$A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B=\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.$$

We know that, the second-order Sturm-Liouville differential expression

$$ly=-(py')'+qy, \tag{1.3}$$

where $p(x)\neq 0$, $\frac{1}{p}, q \in L(0,1)$, with the following boundary condition:

$$\begin{cases} a_{11}y(0)+a_{12}y'(0)+b_{11}y(1)+b_{12}y'(1)=0, \\ a_{21}y(0)+a_{22}y'(0)+b_{21}y(1)+b_{22}y'(1)=0, \end{cases} \tag{1.4}$$

when the coefficient matrices satisfy

$$AQ(0)^{-1}A^*=BQ(1)^{-1}B^*, \tag{1.5}$$

generates a self-adjoint operator, where

$$A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B=\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \quad Q(x)=\begin{pmatrix} 0 & -p(x) \\ p(x) & 0 \end{pmatrix}.$$

This result has great significance in the spectral theory of Sturm-Liouville operator.

A nature thought is to imitate the second-order Sturm-Liouville problem, and thereby the self-adjoined description of the fractional Sturm-Liouville problem is given. However, a fractional derivation operator is quite different from an ordinary differential operator in some properties, especially we fail to get the existence and uniqueness of the fractional equation till now. Therefore, the method can not be completely applied to study our problem.

2 Preliminaries

We will use the following properties of fractional integrals and derivatives.

Definition 2.1[3] For given α with $R(\alpha)>0$, the left and right Riemann-Liouville integrals of order α are defined as

$$(I_{0+}^{\alpha}f)(x)=\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}f(s)\mathrm{d}s, \quad x\in(0,1],$$
$$(I_{1-}^{\alpha}f)(x)=\frac{1}{\Gamma(\alpha)}\int_x^1 (s-x)^{\alpha-1}f(s)\mathrm{d}s, \quad x\in[0,1).$$

Definition 2.2[3] Let $R(\alpha)\in(n-1,n)$, where $n\in\mathbb{N}$, $D=\frac{\mathrm{d}}{\mathrm{d}x}$. The left and right Riemann-Liouville derivatives of order α are defined as