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SOME LIMIT PROPERTIES AND THE GENERALIZED AEP THEOREM FOR NONHOMOGENEOUS MARKOV CHAINS*†

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Abstract

Let $(\xi_n)_{n=0}^{\infty}$ be a Markov chain with the state space $\mathcal{X} = \{1, 2, \cdots, b\}$, $(g_n(x, y))_{n=1}^{\infty}$ be functions defined on $\mathcal{X} \times \mathcal{X}$, and

$$F_{m_n,b_n}(\omega) = \frac{1}{b_n} \sum_{k=m_n+1}^{m_n+b_n} g_k(\xi_{k-1}, \xi_k).$$

In this paper the limit properties of $F_{m_n,b_n}(\omega)$ and the generalized relative entropy density $f_{m_n,b_n}(\omega) = -(1/b_n) \log p(\xi_{m_n,m_n+b_n})$ are discussed, and some theorems on a.s. convergence for $(\xi_n)_{n=0}^{\infty}$ and the generalized Shannon-McMillan (AEP) theorem on finite nonhomogeneous Markov chains are obtained.

Keywords AEP; nonhomogeneous Markov chains; limit theorem; generalized relative entropy density

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1 Introduction

Throughout this paper, let the random variables $(\xi_n)_{n=0}^{\infty}$ be defined on a fixed probability space $(\Omega, \mathcal{F}, P)$ taking on values in a finite set $\mathcal{X} = \{1, 2, \cdots, b\}$. Given two integers, we denote by $\xi_{m,n}$ the random vector of $(\xi_m, \cdots, \xi_n)$ and by $x_{m,n} = (x_m, \cdots, x_n)$ a realization of $\xi_{m,n}$. Suppose the joint distribution of $\xi_{m,n}$ is

$$P(\xi_{m,n} = x_{m,n}) = p(x_{m,n}) > 0, \quad x_i \in \mathcal{X},\ m \leq i \leq n.$$

In what follows we shall assume that $(m_n)_{n=0}^{\infty}$ is a fixed sequence of positive integers, $(b_n)_{n=0}^{\infty}$ is a sequence of integers satisfying: For every $\varepsilon > 0$, $\sum\limits_{n=0}^{\infty} \exp(-\varepsilon b_n) < \infty$.

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Let

$$f_{m_n,b_n}(\omega) = -\frac{1}{b_n}\log p(\xi_{m_n,m_n+b_n}), \tag{1.1}$$

where log is the natural logarithm. The defined quantity of $f_{m_n,b_n}(\omega)$ is referred to as generalized relative entropy density of $(\xi_n)_{n=0}^{\infty}$ (see Wang and Yang [13]). If $(\xi_n)_{n=0}^{\infty}$ is a nonhomogeneous Markov chain with the state space $\mathcal{X} = \{1, 2, \cdots, b\}$, the initial distribution

$$(p(1), \cdots, p(b)), \quad p(i) > 0, \quad i \in \mathcal{X}, \tag{1.2}$$

and the transition matrices

$$P_n = (p_n(j|i))_{b\times b}, \quad p_n(j|i) > 0, \quad i, j \in \mathcal{X},\ n \geq 1, \tag{1.3}$$

then

$$p(x_{m_n,m_n+b_n}) = P(\xi_{m_n,m_n+b_n} = x_{m_n,m_n+b_n}) = p_{m_n}(x_{m_n}) \prod_{k=m_n+1}^{m_n+b_n} p_k(x_k|x_{k-1}),$$

$$f_{m_n,b_n}(\omega) = -\frac{1}{b_n}\left[\log p_{m_n}(\xi_{m_n}) + \sum_{k=m_n+1}^{m_n+b_n} \log p_k(\xi_k|\xi_{k-1})\right], \tag{1.4}$$

where $p_{m_n}(x_{m_n}) = P(\xi_{m_n} = x_{m_n})$, $p_n(j|i) = P(\xi_n = j|\xi_{n-1} = i)$.

The statement of convergence of the relative entropy density $f_{0,n}(\omega)$ to a constant limit called the entropy rate of the process is known as the ergodic theorem of information theory or asymptotic equipartition property (AEP). Shannon [11] first showed that for the stationary ergodic Markov chain $f_{0,n}(\omega)$ converges in probability to a constant. McMillan [7] and Breiman [2] proved, respectively, that if $(\xi_n)_{n=0}^{\infty}$ is stationary and ergodic, then $f_{0,n}(\omega)$ converges in $\mathcal{L}_\infty$ and almost everywhere to a constant. Since then, numerous extension have been made in many directions, such as weakening the hypothesis on the reference measure, state space, index set and required properties of the process. For example, in Feinstein [5], Chung [4], Moy [8], Kiefer [6], Perez [9] and Barron [1].

In the paper of Mark Schwartz [10], it is shown that if $(m_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are two sequences of positive integers, and a measure-preserving ergodic transformation τ, the moving averages $T_n(f) = b_n^{-1}\sum\limits_{k=m_n+1}^{m_n+b_n} f(\tau^k)$ converge a.s.. Motivated by the work of Schwartz, in this paper we first establish a class of limit theorems for finite nonhomogeneous Markov chains, then give an extend Shannon-McMillan (AEP) theorem. The conditions of our main theorems are slightly weaker than those of [13].