Ann. of Appl. Math.
34:4(2018), 416-438

NEW PROOFS OF THE DECAY ESTIMATE WITH SHARP RATE OF THE GLOBAL WEAK SOLUTION OF THE n-DIMENSIONAL INCOMPRESSIBLE NAVIER-STOKES EQUATIONS *

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Abstract

Consider the Cauchy problem for the n-dimensional incompressible Navier-Stokes equations: $\frac{\partial}{\partial t}\mathbf{u}-\alpha\triangle\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla p=\mathbf{f}(\mathbf{x},t)$, with the initial condition $\mathbf{u}(\mathbf{x},0)=\mathbf{u}_0(\mathbf{x})$ and with the incompressible conditions $\nabla\cdot\mathbf{u}=0$, $\nabla\cdot\mathbf{f}=0$ and $\nabla\cdot\mathbf{u}_0=0$. The spatial dimension $n\geq 2$.

Suppose that the initial function $\mathbf{u}_0\in L^1(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$ and the external force $\mathbf{f}\in L^1(\mathbb{R}^n\times\mathbb{R}^+)\cap L^1(\mathbb{R}^+,L^2(\mathbb{R}^n))$. It is well known that there holds the decay estimate with sharp rate: $(1+t)^{1+n/2}\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\leq C$, for all time $t>0$, where the dimension $n\geq 2$, $C>0$ is a positive constant, independent of $\mathbf{u}$ and $(\mathbf{x},t)$.

The main purpose of this paper is to provide two independent proofs of the decay estimate with sharp rate, both are complete, systematic, simplified proofs, under a weaker condition on the external force. The ideas and methods introduced in this paper may have strong influence on the decay estimates with sharp rates of the global weak solutions or the global smooth solutions of similar equations, such as the n-dimensional magnetohydrodynamics equations, where the dimension $n\geq 2$.

Keywords n-dimensional incompressible Navier-Stokes equations; global weak solution; decay estimate with sharp rate; Fourier transformation; Parseval's identity; Gronwall's inequality

2000 Mathematics Subject Classification 35Q20

1 Introduction

1.1 The mathematical model equations

Consider the Cauchy problem for the n-dimensional incompressible Navier-Stokes equations

*Manuscript received September 6, 2018
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$$\frac{\partial}{\partial t}\mathbf{u} - \alpha\triangle\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f}(\mathbf{x},t), \quad \nabla\cdot\mathbf{u} = 0,\ \nabla\cdot\mathbf{f} = 0, \tag{1}$$

$$\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}), \quad \nabla\cdot\mathbf{u}_0 = 0. \tag{2}$$

The real vector valued function $\mathbf{u} = \mathbf{u}(\mathbf{x},t)$ represents the velocity of the fluid at position $\mathbf{x}$ and time t. The real scalar function $p = p(\mathbf{x},t)$ represents the pressure of the fluid at $\mathbf{x}$ and t. The positive constant $\alpha > 0$ represents the diffusion coefficient.

1.2 Previous related results

The existence of a global weak solution or a global smooth solution: Suppose that the initial function $\mathbf{u}_0 \in L^2(\mathbb{R}^n)$ and the external force $\mathbf{f} \in L^1(\mathbb{R}^+, L^2(\mathbb{R}^n))$. Then there exists a global weak solution $\mathbf{u} \in L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^n))$, such that $\nabla\mathbf{u} \in L^2(\mathbb{R}^+, L^2(\mathbb{R}^n))$. Moreover, for the spatial dimension $n = 2$, if the initial function $\mathbf{u}_0 \in L^1(\mathbb{R}^2) \cap H^{2m+1}(\mathbb{R}^2)$ and the external force $\mathbf{f} \in L^1(\mathbb{R}^2 \times \mathbb{R}^+) \cap L^1(\mathbb{R}^+, L^2(\mathbb{R}^2)) \cap L^2(\mathbb{R}^+, H^{2m}(\mathbb{R}^2))$, then there exists a global smooth solution $\mathbf{u} \in L^\infty(\mathbb{R}^+, H^{2m+1}(\mathbb{R}^2))$, such that $\nabla\mathbf{u} \in L^2(\mathbb{R}^+, H^{2m+1}(\mathbb{R}^2))$, for any positive integer $m \geq 1$. Under additional conditions, the solution is infinitely smooth: $\mathbf{u} \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^+)$. There holds the following solution representation

$$\begin{aligned}
\mathbf{u}(\mathbf{x},t) = {} & \frac{1}{(4\pi\alpha t)^{n/2}} \int_{\mathbb{R}^n} \exp\left[-\frac{|\mathbf{x}-\mathbf{y}|^2}{4\alpha t}\right] \mathbf{u}_0(\mathbf{y})\mathrm{d}\mathbf{y} \\
& + \int_0^t \frac{1}{[4\pi\alpha(t-\tau)]^{n/2}} \left\{\int_{\mathbb{R}^n} \exp\left[-\frac{|\mathbf{x}-\mathbf{y}|^2}{4\alpha(t-\tau)}\right] \mathbf{f}(\mathbf{y},\tau)\mathrm{d}\mathbf{y}\right\} \mathrm{d}\tau \\
& - \int_0^t \frac{1}{[4\pi\alpha(t-\tau)]^{n/2}} \left\{\int_{\mathbb{R}^n} \exp\left[-\frac{|\mathbf{x}-\mathbf{y}|^2}{4\alpha(t-\tau)}\right] (\mathbf{u}\cdot\nabla)\mathbf{u}(\mathbf{y},\tau)\mathrm{d}\mathbf{y}\right\} \mathrm{d}\tau \\
& - \int_0^t \frac{1}{[4\pi\alpha(t-\tau)]^{n/2}} \left\{\int_{\mathbb{R}^n} \exp\left[-\frac{|\mathbf{x}-\mathbf{y}|^2}{4\alpha(t-\tau)}\right] \nabla p(\mathbf{y},\tau)\mathrm{d}\mathbf{y}\right\} \mathrm{d}\tau,
\end{aligned}$$

where

$$p(\mathbf{x},t) = (-\triangle)^{-1} \sum_{k=1}^n \sum_{l=1}^n \frac{\partial^2}{\partial x_k \partial x_l}[u_k(\mathbf{x},t)u_l(\mathbf{x},t)],$$

for all $\mathbf{x} \in \mathbb{R}^n$ and $t > 0$.

The decay estimates with sharp rates: Suppose that the initial function $\mathbf{u}_0 \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and the external force $\mathbf{f} \in L^1(\mathbb{R}^n \times \mathbb{R}^+) \cap L^1(\mathbb{R}^+, L^2(\mathbb{R}^n))$. Then there holds the following decay estimate with sharp rate

$$(1+t)^{1+n/2} \int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} \leq C,$$

for all $t > 0$, where $C > 0$ is a positive constant, independent of $\mathbf{u}$ and $(\mathbf{x},t)$.