Ann. of Appl. Math.
32:1(2016), 20-29

SMITH NORMAL FORMAL OF DISTANCE MATRIX OF BLOCK GRAPHS*†

Jing Chen[1,2]‡, Yaoping Hou[2]

(1. *The Center of Discrete Math., Fuzhou University, Fujian 350003, PR China*;
2. *School of Math., Hunan First Normal University, Hunan 410205, PR China*)

Abstract

A connected graph, whose blocks are all cliques (of possibly varying sizes), is called a block graph. Let $D(G)$ be its distance matrix. In this note, we prove that the Smith normal form of $D(G)$ is independent of the interconnection way of blocks and give an explicit expression for the Smith normal form in the case that all cliques have the same size, which generalize the results on determinants.

Keywords block graph; distance matrix; Smith normal form

2000 Mathematics Subject Classification 05C50

1 Introduction

Let G be a connected graph (or strong connected digraph) with vertex set $\{1, 2, \cdots, n\}$. The *distance matrix* $D(G)$ is an $n \times n$ matrix in which $d_{i,j} = d(i, j)$ denotes the distance from vertex i to vertex j. Like the adjacency matrix and Laplacian matrix of a graph, $D(G)$ is also an integer matrix and there are many results on distance matrices and their applications.

For distance matrices, Graham and Pollack [10] proved a remarkable result that gives a formula of the determinant of the distance matrix of a tree depending only on the number n of vertices of the tree. The determinant is given by $\det D = (-1)^{n-1}(n-1)2^{n-2}$. This result has attracted much interest in algebraic graph theory. Graham, Hoffman and Hosoya [8] showed that the determinant of the distance matrix $D(G)$ of a strong connected directed graph G is a function of the distance matrix of its strong blocks:

Theorem 1.1 *If G is a strong connected digraph with strong blocks $G_1, G_2, \cdots, G_r$, then*

*This project was supported by the National Natural Science Foundation of China (Nos. 11501188, 11326057, 11171102) and by Aid program for Science and Technology Innovative Research Team in Higher Educational Institutions of Hunan Province.

†Manuscript received September 6, 2015

‡Corresponding author. E-mail: chenjing827@126.com

$$Cof(D(G)) = \prod_{i=1}^{r} Cof(D(G_i)), \quad and \quad \det(D(G)) = \sum_{i=1}^{r} \det(D(G_i)) \prod_{j \neq i} Cof(D(G_j)),$$

where $Cof(A)$ is the sum of all cofactors of matrix A.

Graham and Lovász [9] computed the inverse of the distance matrix of a tree and studied the characteristic polynomial of the distance matrix of a tree. For more details about the distance matrix spectrum see [16] as well as the references therein. Almost all results obtained for the distance matrix of trees were extended to the case of weighted trees by Bapat et al. [2], and extended to the case that all blocks are cliques in [5,19]. Extensions were done not only concerning the class of graphs but also regarding the distance matrix itself. Indeed, Bapat et al. [4] generalized the concept and its properties of the distance matrix to q-analogue of the distance matrix. Aouchiche and Hansen [1] investigated the spectrum of two distance Laplacian matrices.

For an $n \times n$ integer matrix A, the *Smith normal form* of A, denoted by $Snf(A)$, is an $n \times n$ diagonal integer matrix

$$S = \text{diag}(s_1, s_2, \cdots, s_n),$$

where $s_1, \cdots, s_n$ are nonnegative integers and $s_i | s_{i+1}$ $(1 \leq i \leq n-1)$ satisfies that there exist invertible integer matrices P, Q such that $PAQ = S$. Since $\det Snf(A) = |\det A|$ and $\text{rank}(Snf(A)) = \text{rank}(A)$, the Smith normal form is more refined invariant than (the absolute value of) the determinant and the rank. The Smith normal form of a matrix has some arithmetic and combinatorial significance and was studied in arithmetic geometry [13], in statistical physics [7] and in combinatorics [3]. There are also interpretations of the critical group in discrete dynamics (chip-firing games and abelian sandpile models [3]). For the Laplacian matrix $L(G)$ of a graph G, the above group has been called the critical group (or sandpile group) of a graph G. And there are a few results on the Smith normal form of Laplacian matrix of a graph [12-14,17]. For the results on the smith normal forms of other matrices of graph, see [6,20,21].

According to Theorem 1.1, the determinant of the distance matrix of graph does not change if the blocks of the graph are reassembled in some other way. Since the Smith normal form of a matrix is a refinement of determinant, the following question naturally arises.

Problem 1.1 Is the Smith normal form of the distance matrix of a connected graph independent of the connection way of its blocks?

In the case of a tree T on n vertices, the blocks are precisely the edge (K_2, the complete graph on two vertices) and $\det D(T) = (-1)^{n-1}(n-1)2^{n-2}$. In [11], it is shown that the Smith normal form of $D(T)$ is $\text{diag}(1, 1, 2, \cdots, 2, 2(n-1))$, which is