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EXISTENCE OF PERIODIC SOLUTIONS OF A CLASS OF SECOND-ORDER NON-AUTONOMOUS SYSTEMS*†

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Abstract

In this paper, we are concerned with the existence of periodic solutions of second-order non-autonomous systems. By applying the Schauder's fixed point theorem and Miranda's theorem, a new existence result of periodic solutions is established.

Keywords second-order non-autonomous systems; periodic solutions; Schauder's fixed point theorem; Miranda's theorem

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1 Introduction

In the past few years, there has been considerable interest in the existence of periodic solutions of the following second-order periodic boundary value problems

$$u''(t) + a(t)u(t) = f(t, u(t)) + c(t),$$
$$u(0) = u(T), \quad u'(0) = u'(T),$$

where $a, c \in L^1(0, T)$ and $f : [0, T] \times \mathbf{R} \to \mathbf{R}$ is continuous. For more details please see [1-6] and the references therein. In particular, many authors mentioned above paid their attention to the non-resonant case, that is, the unique solution of the following linear problem

$$u''(t) + a(t)u(t) = 0, \quad u(0) = u(T),\ u'(0) = u'(T) \tag{1.1}$$

is the trivial one. To the end, the function a is supposed to satisfy the basic assumption:

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(H0) The Green's function $G(t,s)$ of the linear problem (1.1) is nonnegative for every $(t,s) \in [0,T] \times [0,T]$. See [3] for the details.

It is well known that if (1.1) is non-resonant and h is a L^1-function then the Fredholm's alternative theorem implies that the nonhomogeneous problem

$$u''(t) + a(t)u(t) = h(t), \quad u(0) = u(T),\ u'(0) = u'(T)$$

always has a unique solution, which can be written as

$$u(t) = \int_0^T G(t,s)h(s)\mathrm{d}s.$$

And consequently, the linear problem (1.1) is non-resonant. On the other hand, several authors have focused their attention to the existence of periodic solutions of the second-order nonlinear systems. Here we refer the readers to Chu, Torres and Zhang [7], Franco and Webb [8], Cao and Jiang [9] and Wang [10]. Especially in [9], Cao and Jiang obtained several existence results of periodic solutions of the following second order coupled systems

$$\begin{cases} u''(t) + a_1(t)u(t) = f_1(t, v(t)) + e_1(t), \\ v''(t) + a_2(t)v(t) = f_2(t, u(t)) + e_2(t), \end{cases} \tag{1.2}$$

where $f, g : (\mathbf{R}/T\mathbf{Z}) \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ are continuous and $a_i, e_i \in C(\mathbf{R}/T\mathbf{Z}, \mathbf{R}),\ i = 1, 2$.

Clearly, the above mentioned papers all dealt with the non-resonant problems, that is $a_i(t) \not\equiv 0,\ i = 1, 2$. Now, the natural question is whether or not there is a periodic solution of (1.2) if $a_i(t) \equiv 0,\ i = 1, 2$?

In this paper, we shall establish a new existence result of periodic solutions of the resonant coupled systems

$$\begin{cases} u''(t) = f(t, u(t), v(t)) + e_1(t), \\ v''(t) = g(t, u(t), v(t)) + e_2(t). \end{cases} \tag{1.3}$$

To the best of our knowledge, the existence results of periodic solutions of the above systems are relatively little, and our result shall fill this gap.

The main result of this paper is as follows.

Theorem 1.1 *Suppose that*

(H1) $f, g \in C((\mathbf{R}/T\mathbf{Z}) \times \mathbf{R} \times \mathbf{R}, \mathbf{R})$ *are bounded. There are two positive constants* l_1 *and* l_2 *such that for each* $(t, x, y) \in \mathbf{R} \times \mathbf{R} \times \mathbf{R}$,

$$\begin{aligned} f(t,x,y)x < 0, &\qquad |x| \geq l_1, \\ g(t,x,y)y < 0, &\qquad |y| \geq l_2; \end{aligned}$$

(H2) $e_i \in C(\mathbf{R}/T\mathbf{Z}, \mathbf{R})$ *with mean value*