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L^6 BOUND FOR BOLTZMANN DIFFUSIVE LIMIT*

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Abstract

We consider diffusive limit of the Boltzmann equation in a periodic box. We establish L^6 estimate for the hydrodynamic part $\mathbf{P}f$ of particle distribution function, which leads to uniform bounds global in time.

Keywords L^6 estimate; Boltzmann equation; diffusive limit
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1 Introduction

We study the diffusive limit of the Boltzmann equation

$$\varepsilon\partial_t F + v\cdot\nabla_x F = \frac{1}{\varepsilon}Q(F,F)$$

with the particle distribution function $F(t,x,v) = \mu+\varepsilon\sqrt{\mu}f(t,x,v)$ in a periodic box of $\mathbf{T}^3\times\mathbf{R}^3$, where $\mu = \frac{1}{\{2\pi\}^{3/2}}\mathrm{e}^{-|v|^2/2}$ is a normalized Maxwellian. For simplicity, we assume the collision operator Q is given by the classical hard-sphere interaction. In terms of perturbation f, we have

$$\varepsilon\partial_t f + v\cdot\nabla_x f + \frac{1}{\varepsilon}Lf = \Gamma(f,f). \tag{1}$$

We denote the hydrodynamic part of $f(t,x,v)$

$$\mathbf{P}f \equiv \Big\{a(t,x) + b(t,x)\cdot v + c(t,x)\Big(\frac{|v|^2-3}{2}\Big)\Big\}\sqrt{\mu}$$

as the L^2_v orthogonal projection of f with respect to $\{1, v, (\frac{|v|^2-3}{2})\}\sqrt{\mu}$. It is well-known that *formally* as $\varepsilon\to 0$,

$$\mathbf{P}f \to \Big\{\rho(t,x) + u(t,x)\cdot v + \theta(t,x)\Big(\frac{|v|^2-3}{2}\Big)\Big\}\sqrt{\mu},$$

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where $[u(t,x),\theta(t,x)]$ satisfies the celebrated incompressible Navier-Fourier system

$$\begin{aligned}&u_t+u\cdot\nabla u+\nabla p=\varkappa\Delta u,\qquad \nabla\cdot u=0,\\&\theta_t+u\cdot\nabla\theta=\kappa\Delta\theta,\end{aligned}$$

with Boussineq approximation $\rho+\theta=0$, see [3].

As in many singular perturbation problems [3], the key is to obtain uniform estimates for solutions to the Boltzmann equation (1). In [3], a nonlinear energy method leads to uniform bounds in high Sobolev norms. A natural question left open was whether one can obtain uniform bounds with lower regularity. This is particularly important in the study of boundary value problem [1,2,4], in which high Sobolev regularity is impossible in general.

As in [1], we establish uniform bounds without any Sobolev regularity in this paper. The main idea is to start with basic energy estimate, which leads to control of the microscopic (kinetic) part

$$||\{\mathbf{I}-\mathbf{P}\}f||_\nu=\sqrt{\int_{\mathbf{T}^3\times\mathbf{R}^3}\nu(v)\{\mathbf{I}-\mathbf{P}\}f^2},$$

where the collision frequency $\nu(v)\backsim\langle v\rangle$, for the hard-sphere case. By the positivity estimate in [4], the macroscopic part $||\mathbf{P}f||_{L^2_{t,x,v}}$ can be controlled. Unfortunately, such a $||\mathbf{P}f||_2$ bound is not strong enough to control the nonlinearity $\Gamma(f,f)$ uniformly in ε. The main novelty is to obtain uniform estimates in ε for $\mathbf{P}f$ with an improved L^6 estimate for the macroscopic part $\mathbf{P}f$. This new estimate leads to an improved L^∞ bound, which completes the control of $\Gamma(f,f)$.

We now define energy $\mathcal{E}(t)$ and dissipation rate $\mathcal{D}(t)$ as

$$\begin{aligned}&\mathcal{E}(t)\equiv||f(t)||_2^2+||f_t(t)||_2^2,\\&\mathcal{D}(t)\equiv\frac{1}{\varepsilon^2}||\{\mathbf{I}-\mathbf{P}\}f(t)||_\nu^2+\frac{1}{\varepsilon^2}||\{\mathbf{I}-\mathbf{P}\}f_t(t)||_\nu^2.\end{aligned}$$

Our main result consists of the following a-priori uniform estimate.

Theorem 1 *Assume hard-sphere collision kernel. Assume f is a solution to the Boltzmann equation* (1). *Let $w=\langle v\rangle^l=\{1+|v|^2\}^l$, some $l\gg 1$,*

$$|||f_0|||\equiv||f_0||_2+||f_{0t}||_2+\frac{1}{\varepsilon}||\{\mathbf{I}-\mathbf{P}\}f_0||_{L^2}+\sqrt{\varepsilon}||wf_0||_\infty+\frac{1}{\varepsilon}||\{\mathbf{I}-\mathbf{P}\}f_{0t}||_{L^2}+\sqrt{\varepsilon}||wf_{0t}||_\infty.$$

If $|||f_0|||\ll 1$, then for any $0\le t\le\infty$,

$$\mathcal{E}(t)+\int_0^t\mathcal{D}(s)\mathrm{d}s+||\mathbf{P}f(t)||_{L^6}+||\mathbf{P}f(t)||_{L^2_tL^2_x}+\varepsilon^{1/2}||wf(t)||_\infty\lesssim|||f_0|||.$$