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THE IMPROVED FOURIER SPLITTING METHOD AND DECAY ESTIMATES OF THE GLOBAL SOLUTIONS OF THE CAUCHY PROBLEMS FOR NONLINEAR SYSTEMS OF FLUID DYNAMICS EQUATIONS*

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Dedicated to Professor Boling Guo on the occasion of his eightieth birthday!

Abstract

Consider the Cauchy problems for an n-dimensional nonlinear system of fluid dynamics equations. The main purpose of this paper is to improve the Fourier splitting method to accomplish the decay estimates with sharp rates of the global weak solutions of the Cauchy problems. We will couple together the elementary uniform energy estimates of the global weak solutions and a well known Gronwall's inequality to improve the Fourier splitting method. This method was initiated by Maria Schonbek in the 1980's to study the optimal long time asymptotic behaviours of the global weak solutions of the nonlinear system of fluid dynamics equations. As applications, the decay estimates with sharp rates of the global weak solutions of the Cauchy problems for n-dimensional incompressible Navier-Stokes equations, for the n-dimensional magnetohydrodynamics equations and for many other very interesting nonlinear evolution equations with dissipations can be established.

Keywords nonlinear systems of fluid dynamics equations; global weak solutions; decay estimates; uniform energy estimates; Fourier transformation; Plancherel's identity; Gronwall's inequality; improved Fourier splitting method

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1 Introduction

1.1 The mathematical model equations

First of all, consider the Cauchy problems for the n-dimensional incompressible Navier-Stokes equations

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$$\frac{\partial \mathbf{u}}{\partial t} - \alpha \triangle \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f}(\mathbf{x}, t), \quad \nabla \cdot \mathbf{u} = 0, \quad \nabla \cdot \mathbf{f} = 0, \tag{1}$$

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \quad \nabla \cdot \mathbf{u}_0 = 0. \tag{2}$$

The real vector valued function $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ represents the velocity of the fluid at position $\mathbf{x}$ and time t. The real scalar function $p = p(\mathbf{x}, t)$ represents the pressure of the fluid at $\mathbf{x}$ and t.

Secondly, consider the Cauchy problems for the n-dimensional magnetohydrodynamics equations

$$\frac{\partial \mathbf{u}}{\partial t} - \frac{1}{\mathrm{RE}} \triangle \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} - (\mathbf{A} \cdot \nabla)\mathbf{A} + \nabla P = \mathbf{f}(\mathbf{x}, t), \tag{3}$$

$$\frac{\partial \mathbf{A}}{\partial t} - \frac{1}{\mathrm{RM}} \triangle \mathbf{A} + (\mathbf{u} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{u} = \mathbf{g}(\mathbf{x}, t), \tag{4}$$

$$\nabla \cdot \mathbf{u} = 0, \quad \nabla \cdot \mathbf{f} = 0, \quad \nabla \cdot \mathbf{A} = 0, \quad \nabla \cdot \mathbf{g} = 0, \tag{5}$$

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \quad \mathbf{A}(\mathbf{x}, 0) = \mathbf{A}_0(\mathbf{x}), \quad \nabla \cdot \mathbf{u}_0 = 0, \quad \nabla \cdot \mathbf{A}_0 = 0. \tag{6}$$

In this system, the real vector valued function $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ represents the velocity of the fluid at position $\mathbf{x}$ and time t, the real vector valued function $\mathbf{A} = \mathbf{A}(\mathbf{x}, t)$ represents the magnetic field at position $\mathbf{x}$ and time t. The real scalar function $P(\mathbf{x}, t) = p(\mathbf{x}, t) + \frac{M^2}{2\mathrm{RE}\cdot\mathrm{RM}} |\mathbf{A}(\mathbf{x}, t)|^2$ represents the total pressure, where the real scalar function $p = p(\mathbf{x}, t)$ represents the pressure of the fluid and $\frac{1}{2}|\mathbf{A}(\mathbf{x}, t)|^2$ represents the magnetic pressure. Additionally, $M > 0$ represents the Hartman constant, RE represents the Reynolds constant and RM represents the magnetic Reynolds constant.

Now let us consider the Cauchy problems for the following n-dimensional nonlinear system of fluid dynamics equations

$$\frac{\partial \mathbf{u}}{\partial t} - \alpha \triangle \mathbf{u} + \mathcal{N}(\mathbf{u}, \nabla \mathbf{u}) = \mathbf{f}(\mathbf{x}, t), \tag{7}$$

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}). \tag{8}$$

In this system, $\alpha > 0$ is a positive constant, $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ represents the spatial variable, the dimension $n \geq 3$. Moreover, $\mathbf{u}(\mathbf{x}, t) = (u_1(\mathbf{x}, t), u_2(\mathbf{x}, t), \cdots, u_m(\mathbf{x}, t))$ represents the unknown function, $\mathbf{f}(\mathbf{x}, t) = (f_1(\mathbf{x}, t), f_2(\mathbf{x}, t), \cdots, f_m(\mathbf{x}, t))$ represents the external force, and $\mathcal{N}(\mathbf{u}, \nabla \mathbf{u}) = (N_1(\mathbf{u}, \nabla \mathbf{u}), N_2(\mathbf{u}, \nabla \mathbf{u}), \cdots, N_m(\mathbf{u}, \nabla \mathbf{u}))$ represents the nonlinear function, which is sufficiently smooth, $m \geq n$ is an integer.

The general system (7)-(8) contains the n-dimensional incompressible Navier-Stokes equations (1)-(2) and the n-dimensional magnetohydrodynamics equations (3)-(6) as particular examples. The general system also contains many other interesting nonlinear evolution equations with dissipations as examples.

Many mathematicians have accomplished the existence of the global weak solutions of the Cauchy problems for the n-dimensional incompressible Navier-Stokes