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Weighted Norm Inequalities for Marcinkiewicz Integrals with Non-Smooth Kernels on Spaces of Homogeneous Type

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Abstract. In this article, we obtain some weighted estimates for Marcinkiewicz integrals with non-smooth kernels on spaces of homogeneous type. The weight w considered here belongs to the Muckenhoupt's class A_∞. Moreover, weighted estimates for commutators of BMO functions and Marcinkiewicz integrals are also given.

Key Words: Commutators, Muckenhoupt weights, Marcinkiewicz integrals, Singular integrals, Sharp maximal functions, BMO functions, Young functions, Luxemburg norm, Spaces of homogeneous type.

AMS Subject Classifications: 42B20, 42B25, 42B35

1 Introduction

Let $(\mathcal{X}, d, \mu)$ be a space of homogeneous type, endowed with a metric distance d on $\mathcal{X} \times \mathcal{X}$ satisfying

$$d(x,z) \leq \kappa\,(d(x,y) + d(y,z)) \text{ for some fixed constant } \kappa \geq 1 \text{ and for all } x,y,z \in \mathcal{X}, \quad (1.1)$$

and a regular Borel measure μ on $\mathcal{X}$ such that the doubling property

$$\mu(B(x;2r)) \leq C\mu(B(x;r)) < \infty \quad (1.2)$$

holds for some fixed constant $C \geq 1$, for all $x \in \mathcal{X}$ and for all $r > 0$, where $B(x;r) = \{y \in \mathcal{X} : d(x,y) < r\}$. The above property implies that there exist some fixed constants $C \geq 1$, $n > 0$ such that

$$\mu(B(x;\lambda r)) \leq C\lambda^n \mu(B(x;r)), \quad (1.3)$$

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uniformly for all $\lambda \geq 1$, $x \in \mathcal{X}$, and $r > 0$. The parameter n measures the "dimension" of the space $\mathcal{X}$. There also exist constants C, N ($C \geq 1, 0 \leq N \leq n$) such that

$$\mu(B(y;r)) \leq C\left(1+\frac{d(x,y)}{r}\right)^N \mu(B(x;r)) \tag{1.4}$$

uniformly for all $x,y \in \mathcal{X}$ and all $r > 0$. The reader can find more information on this subject in [2,3].

Let T be a bounded linear operator on $L^2(\mathcal{X})$ with an associated kernel $K(x,y)$ in the sense that

$$Tf(x) = \int_{\mathcal{X}} K(x,y)f(y)d\mu(y), \tag{1.5}$$

where f is a continuous function with compact support, $x \notin \mathrm{supp} f$; and $K(x,y)$ is a measurable function defined on $(\mathcal{X}\times\mathcal{X})\diagdown\Delta$ with $\Delta = \{(x,x) : x \in \mathcal{X}\}$.

The authors in [4,6] assumed that there exists a class of operators A_t $(t > 0)$ which can be represented by the kernels $a_t(x,y)$ in the sense that

$$A_t u(x) = \int_{\mathcal{X}} a_t(x,y)u(y)d\mu(y) \quad \text{for every function } \ u \in L^1(\mathcal{X}) \cap L^2(\mathcal{X}).$$

Moreover, the kernels $a_t(x,y)$ satisfy the following conditions

$$|a_t(x,y)| \leq h_t(x,y) \quad \text{for all } \ x,y \in \mathcal{X}, \tag{1.6a}$$

where $h_t(x,y) = (\mu(B(x;t^{1/m})))^{-1}s((d(x,y))^m t^{-1})$ for some positive constant m. (1.6b)

Here s is a positive, bounded, decreasing function satisfying

$$\lim_{r\to\infty} r^{n+\sigma}s(r^m) = 0 \tag{1.7}$$

for some $\sigma > N$, where n and N appear in (1.3) and (1.4) respectively.

Remark 1.1. The functions h_t above satisfy the following properties (see [5,6]):

1) There exist positive constants C_1 and C_2 such that

$$C_1 \leq \int_{\mathcal{X}} h_t(x,y)d\mu(x) \leq C_2 \ \text{ uniformly in } \ t \ \text{ and } \ dy.$$

2) There exists a positive constant C such that

$$\int_{\mathcal{X}} h_t(x,y)|f(x)|d\mu(x) \leq C\mathcal{M}f(y) \quad \text{and} \quad \int_{\mathcal{X}} h_t(x,y)|f(y)|d\mu(y) \leq C\mathcal{M}f(x).$$

Here $\mathcal{M}f(x)$, the Hardy-Littlewood maximal function, is defined by

$$\mathcal{M}f(x) = \sup_{B\ni x}\left\{\frac{1}{\mu(B)}\int_B |f(y)|d\mu(y)\right\},$$

where the supremum is taken over all balls B containing x.