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Multistationarity of Reaction Networks with One-Dimensional Stoichiometric Subspaces

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Abstract. We study the multistationarity for the reaction networks with one dimensional stoichiometric subspaces, and we focus on the networks admitting finitely many positive steady states. We provide a necessary condition for a network to admit multistationarity in terms of the stoichiometric coefficients, which can be described by "arrow diagrams". This necessary condition is not sufficient unless there exist two reactions in the network such that the subnetwork consisting of the two reactions admits at least one and finitely many positive steady states. We also prove that if a network admits at least three positive steady states, then it contains at least three "bi-arrow diagrams". More than that, we completely characterize the bi-reaction networks that admit at least three positive steady states.

AMS subject classifications: 92C40, 92C45

Key words: Reaction networks, mass-action kinetics, multistationarity, multistability.

1 Introduction

This work is motivated by the multistability problem of dynamical systems arising from biochemical reaction networks (under mass-action kinetics): for which rate constants, a network has at least two stable positive steady states in the same stoichiometric compatibility class? Multistability is a frontier topic in mathematical biology since it widely exists in the decision-making process and switch-like behavior in cellular signaling (e.g., [1,8,10,17,27]). Multistability problem is known to be a special real quantifier elimination problem so it is challenging to solve it by the computational tools in real algebraic geometry (e.g., [4,12]). Given a network, one way to find multistability is to look for a witness for (nondegenerate) multistationarity, i.e., a choice of parameters (rate constants and total constants) such that the network has at least two positive (nondegenerate) steady states.

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In practice, if the number of positive nondegenerate steady states is large enough, one can usually obtain at least two stable ones (e.g., [11, 18, 25]). Deciding multistationarity or computing the witnesses for multistationarity is not easy neither but there exists a collection of nice methods (e.g., [6, 13, 16, 21]). For instance, one typical approach is to check if the determinant of a certain Jacobian matrix changes sign [2, 5, 7, 26]. Especially, for the networks with binomial steady-state equations, deciding multistationary can be unexpectedly simple [9, 20].

Since multistability or nondegenerate multistationarity can be lifted from small subnetworks to the corresponding large networks [3, 14], criteria of (nondegenerate) multistationarity for small networks with only one species or up to two reactions (possibly reversible) are studied in [15, 22]. In [15, 22], the authors completely characterized one-species networks by "arrow diagrams". For instance, the network "$X_1 \to 2X_1$, $2X_1 \to X_1$" can be described as the arrow diagram $(\to,\leftarrow)$ (see Definition 4.3). At the end of [15], the authors wonder if their results can be extended to more general networks with one-dimensional stoichiometric subspaces (note here, for a network with two reactions, if it admits multistationarity, then it has a one-dimensional stoichiometric subspace [15]). More specifically, they proposed the following question:

Question 1.1. [15, Question 6.1] Consider a network G with a one-dimensional stoichiometric subspace. For G to be multistationary, is it necessary for G to have an embedded one-species network with arrow diagram $(\to,\leftarrow)$ and another with arrow diagram $(\leftarrow,\to)$? Is it sufficient?

It is remarkable that for the networks with one-dimensional stoichiometric subspaces, multistationarity is equivalent to nondegenerate multistationarity if the maximum number of positive steady states is finite (see Theorem 2.1). It is also worth mentioning that if a network with a one-dimensional stoichiometric subspace admits multistability, then it admits at least three positive steady states (e.g., [24, Theorem 3.4]). So, it is also important to extend the results in [15] to the networks admitting at least three positive steady states.

In this paper, we study the multistationarity problem for the networks with one-dimensional stoichiometric subspaces, and we focus on the networks admitting finitely many positive steady states. We answer Question 1.1 and extend the problem by the following results.

(1) If a network admits multistationarity, then the network has an embedded one-species network with arrow diagram $(\to,\leftarrow)$ and another with arrow diagram $(\leftarrow,\to)$ (Theorem 4.1). The converse is also true if we additionally assume that a subnetwork consisting of two reactions from the original network admits at least one and finitely many positive steady states (Theorem 4.2).

(2) If a network admits at least three positive steady states, then it contains at least three bi-arrow diagrams (Theorem 5.1 and Corollary 5.2).

(3) We completely characterize the stoichiometric coefficients of the bi-reaction networks that admit at least three positive steady states (Theorem 6.1).