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A Dual-Horizon Nonlocal Diffusion Model and Its Finite Element Discretization

Mingchao Bi[1], Lili Ju[2] and Hao Tian[1,*]

[1] *School of Mathematical Sciences, Ocean University of China, Qingdao, Shandong 266100, China*
[2] *Department of Mathematics, University of South Carolina, Columbia, SC 29208, USA*

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Abstract. In this paper, we present a dual-horizon nonlocal diffusion model, in which the influence area at each point consists of a standard sphere horizon and an irregular dual horizon and its geometry is determined by the distribution of the varying horizon parameter. We prove the mass conservation and maximum principle of the proposed nonlocal model, and establish its well-posedness and convergence to the classical diffusion model. Noticing that the dual horizon-related term in fact vanishes in the corresponding variational form of the model, we then propose a finite element discretization for its numerical solution, which avoids the difficulty of accurate calculations of integrals on irregular intersection regions between the mesh elements and the dual horizons. Various numerical experiments in two and three dimensions are also performed to illustrate the usage of the variable horizon and demonstrate the effectiveness of the numerical scheme.

AMS subject classifications: 65R20, 65N35, 74S05

Key words: Nonlocal diffusion, dual-horizon, maximum principle, finite element discretization, asymptotically compatible.

1. Introduction

In recent years, a number of nonlocal models in the form of integral-type formulations rather than partial differential equations have been developed, which allow them to reduce the regularity requirements and capture long-range interactions at different scales. Nonlocal continuum models can provide a more accurate description for a broad spectrum of applications from fracture mechanics [1, 24, 25, 28, 29, 33–36, 44], diffusion process [27, 32], image analysis [3, 21, 22], biology [30], phase transition [12, 15]

*Corresponding author. *Email addresses:* `bmc@stu.ouc.edu.cn` (M. Bi), `ju@math.sc.edu` (L. Ju), `haot@ouc.edu.cn` (H. Tian)

to mathematical analysis and computational simulations [2,4–11,13,14,16–20,23,26, 41–43,45–47]. Among these, we are particularly interested in the nonlocal diffusion models taking the following form:

$$-\int_{\mathcal{B}(\boldsymbol{x},\delta)} \rho_\delta\big(\|\boldsymbol{y}-\boldsymbol{x}\|\big)\big(u(\boldsymbol{y})-u(\boldsymbol{x})\big)\, d\boldsymbol{y} = f(\mathbf{x}), \tag{1.1}$$

where u is the unknown function, f is the source function, $\mathcal{B}(\boldsymbol{x},\delta)$ is a neighborhood of $\mathbf{x}$, named as the horizon with δ denoting its sizing parameter, and $\rho_\delta(\|\boldsymbol{y}-\boldsymbol{x}\|)$ is a nonnegative symmetric weight function with $\|\cdot\|$ denoting the Euclidean metric. The horizon parameter δ has been usually treated as a constant until recently. Basically there are two major motivations to take a spatially variable horizon. The first is for the sake of computational efficiency, it is often necessary to vary the horizon sizes with respect to the spatial distribution of the material points, e.g. for adaptive refinement. In the implementation of traditional nonlocal diffusion models with constant horizon radius, the horizon has to be determined with respect to the lowest requirement of resolution in order to achieve acceptable accuracy [31]. The other is to allow the horizon parameter to vanish as the material points approach to the boundary, so that the traditional boundary condition could be directly imposed to the nonlocal diffusion models [37,40].

In the past few years, some nonlocal models with variable horizons have been developed, including the dual-horizon peridynamics for nonlocal mechanics [31] and nonlocal diffusion models allowing heterogeneous localization [40]. In [31] the ideal of dual-horizon was firstly proposed in the area of peridynamics, which is derived based on the Newton's third law. In this model, the influence area for a point includes both a traditional sphere horizon and a dual-horizon, and the geometry of the dual-horizon can be highly irregular depending on the distribution of the horizon at each point. The dual-horizon peridynamics is discretized using the meshfree method in [31], and with such a discretization scheme, the quadratures on the intersection areas between the mesh elements and the horizons/the dual-horizons must be calculated. On the other hand, due to the high irregularity of these regions, these integrals on dual horizons are quite hard be calculated with high accuracy, especially for high dimensional problems. In [37,40], nonlocal diffusion models with a shrinking horizon as it approaches the inner interface (where the nonlocal models and local models connect) are proposed, as well as the weak formulation of the nonlocal models with variable horizon. In addition, the finite element discretization is used for their numerical approximation and some numerical examples of one-dimension problems are presented.

In this paper, we propose a dual-horizon nonlocal diffusion model. The basic structure of our model is consistent with the dual-horizon peridynamics proposed in [31], in which the integral equation include two parts, one is with the traditional sphere horizon $B(\mathbf{x},\delta(\mathbf{x}))$ and the other is with an irregular dual horizon $\widetilde{B}(\mathbf{x},\delta(\mathbf{y}))$. We present rigorous mathematical analysis on the proposed nonlocal model and its numerical discretization. We prove the mass conservation, the maximum principle and the well-posedness of the model, in addition to the convergence to the local classic diffusion