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An Extension of the Order-Preserving Mapping to the WENO-Z-Type Schemes

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Abstract. In the present study, we extend the order-preserving (OP) criterion proposed in our latest studies to the WENO-Z-type schemes. Firstly, we innovatively present the concept of the generalized mapped WENO schemes by rewriting the Z-type weights in a uniform formula from the perspective of the mapping relation. Then, we naturally introduce the OP criterion to improve the WENO-Z-type schemes, and the resultant schemes are denoted as MOP-GMWENO-X, where the notation "X" is used to identify the version of the existing WENO-Z-type scheme in this paper. Finally, extensive numerical experiments have been conducted to demonstrate the benefits of these new schemes. We draw the conclusion that, the convergence properties of the proposed schemes are equivalent to the corresponding WENO-X schemes. The major benefit of the new schemes is that they have the capacity to achieve high resolutions and simultaneously remove spurious oscillations for long simulations. The new schemes have the additional benefit that they can greatly decrease the post-shock oscillations on solving 2D Euler problems with strong shock waves.

AMS subject classifications: 65M06, 65M12

Key words: WENO, Z-type weights, order-preserving generalized mapping, hyperbolic systems.

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1 Introduction

Over the past several decades, the WENO methods [5–7, 22–28] have received considerable scholarly attention. The first WENO scheme that can obtain the designed convergence order of accuracy was proposed by Jiang and Shu [22], dubbed WENO-JS. By using the information of all r-point substencils of the ENO scheme [1–4], WENO-JS maintains the ENO property near the region with discontinuities or large gradients and in the

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meantime achieves the designed convergence rates of accuracy. It was pointed out by Henrick et al. in [29] that the fifth-order WENO-JS scheme can not recover the designed accuracy at critical points of order $n_{\rm cp}=1$. For function g, one has $g'=0$, $g''\neq 0$ for $n_{\rm cp}=1$. Similarly, $g'=0$, $g''=0$, $g'''\neq 0$ for $n_{\rm cp}=2$, etc. Then, the sufficient condition for optimality of the convergence rates of accuracy was derived in [33], and this sufficient condition can be extended to higher order cases trivially [30]. In the work of Henrick et al. [29], a mapping function, namely $(g^{\rm M})_s(\omega^{\rm JS})$, was designed and the resultant mapped WENO scheme, dubbed WENO-M, can achieve the designed convergence properties even in the presence of critical points. It is since the work of Henrick et al. [29] that the study of different mapped WENO methods has gained momentum, and a series of new mapping functions [30–32, 37, 38, 44] have been proposed by obeying the similar principles proposed by Henrick et al. [29].

Later, the work of Henrick et al. [29] inspired the development of a new family of nonlinear weights, dubbed Z-type weights. From a different perspective, Borges et al. [33] proposed another version of nonlinear weights by using available and previously unused information of the WENO-JS scheme. In other words, a global smoothness indicator (GSI) of higher order, obtained via a linear combination of the original smoothness indicators of the WENO-JS scheme, was proposed and employed to devise the new nonlinear weights. The resultant scheme was denoted as WENO-Z. Because of the success of the WENO-Z scheme that its nonlinear weights can satisfy the sufficient conditions for optimality of the convergence order without any costly mapping processes, leading to superior results with almost the same computational effort of the WENO-JS method, different researchers have developed a multitude of techniques to design their Z-type weights [8–11, 14, 19–21, 39, 45, 46] by obeying the similar principles proposed by Borges et al. [33]. In this paper, all the WENO schemes using Z-type weights is collectively called WENO-Z-type schemes. We will give a brief review of several WENO-Z-type schemes in subsection 2.3.

Despite the success mainly for short-output-time simulations, the family of mapped WENO schemes has a serious and ubiquitous problem in calculations with long output times, that is, they can hardly avoid spurious oscillations and meanwhile preserve high resolutions for long simulations. This disadvantage in long simulations of the mapped WENO methods was firstly noticed and carefully studied by Feng et al. [31] and it has attracted considerable attention over the past decade [30, 32, 34, 36, 37, 42, 43]. However, up to now, far too little attention has been paid to the long-output-time simulations of the WENO-Z-type schemes. Indeed, our extensive calculations (see Subsection 4.3 below) show that the WENO-Z-type schemes also terribly suffer from either losing high resolutions or generating numerical oscillations on long-run calculations. Nevertheless, undoubtedly, this issue is worthy of scholarly attention.

Accordingly, in this article, we would like to focus on the theme of addressing the aforementioned drawback of the WENO-Z-type schemes. First of all, we give the important observation and analysis of the implicit relationship between the nonlinear weights of the WENO-JS scheme and the Z-type weights, denoted as IMR (standing for *implicit*