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Deep Ritz Methods for Laplace Equations with Dirichlet Boundary Condition

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Abstract. Deep Ritz methods (DRM) have been proven numerically to be efficient in solving partial differential equations. In this paper, we present a convergence rate in H^1 norm for deep Ritz methods for Laplace equations with Dirichlet boundary condition, where the error depends on the depth and width in the deep neural networks and the number of samples explicitly. Further we can properly choose the depth and width in the deep neural networks in terms of the number of training samples. The main idea of the proof is to decompose the total error of DRM into three parts, that is approximation error, statistical error and the error caused by the boundary penalty. We bound the approximation error in H^1 norm with ReLU^2 networks and control the statistical error via Rademacher complexity. In particular, we derive the bound on the Rademacher complexity of the non-Lipschitz composition of gradient norm with ReLU^2 network, which is of immense independent interest. We also analyze the error inducing by the boundary penalty method and give a prior rule for tuning the penalty parameter.

AMS subject classifications: 62G05, 65N12, 65N15, 68T07

Key words: Deep Ritz methods, convergence rate, Dirichlet boundary condition, approximation error, Rademacher complexity.

1 Introduction

Partial differential equations (PDEs) are one of the fundamental mathematical models in studying a variety of phenomenons arising in science and engineering. There have been

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established many conventional numerical methods successfully for solving PDEs in the case of low dimension $(d \leq 3)$, particularly the finite element method [8,9,25,37,44]. However, one will encounter some difficulties in both of theoretical analysis and numerical implementation when extending conventional numerical schemes to high-dimensional PDEs. The classic analysis of convergence, stability and any other properties will be trapped into troublesome situation due to the complex construction of finite element space [8, 9]. Moreover, in the term of practical computation, the scale of the discrete problem will increase exponentially with respect to the dimension.

Motivated by the well-known fact that deep learning method for high-dimensional data analysis has been achieved great successful applications in discriminative, generative and reinforcement learning [18, 22, 42], solving high dimensional PDEs with deep neural networks becomes an extremely potential approach and has attracted much attentions [2, 6, 21, 31, 38, 43, 48, 50]. Roughly speaking, these works can be divided into three categories. The first category is using deep neural network to improve classical numerical methods, see for example [19, 24, 45, 47]. In the second category, the neural operator is introduced to learn mappings between infinite-dimensional spaces with neural networks [1, 28, 29]. For the last category, one utilizes deep neural networks to approximate the solutions of PDEs directly including physics-informed neural networks (PINNs) [38], deep Ritz method (DRM) [48] and weak adversarial networks (WAN) [50]. PINNs is based on residual minimization for solving PDEs [2, 31, 38, 43]. Proceed from the variational form, [48–50] propose neural-network based methods related to classical Ritz and Galerkin method. In [50], WAN are proposed inspired by Galerkin method. Based on Ritz method, [48] proposes the DRM to solve variational problems corresponding to a class of PDEs.

1.1 Related works and contributions

The idea using neural networks to solve PDEs goes back to 1990's [12,26]. Although there are great empirical achievements in recent several years, a challenging and interesting question is to provide a rigorous error analysis such as finite element method. Several recent efforts have been devoted to making processes along this line, see for example [14, 15, 23, 30, 32, 34, 36, 41, 46]. In [32], least squares minimization method with two-layer neural networks is studied, the optimization error under the assumption of over-parametrization and generalization error without the over-parametrization assumption are analyzed. In [30, 49], the generalization error bounds of two-layer neural networks are derived via assuming that the exact solutions lie in spectral Barron space.

Dirichlet boundary condition corresponds to a constrained minimization problem, which may cause some difficulties in computation. The penalty method has been applied in finite element methods and finite volume method [4, 33]. It is also been used in deep PDEs solvers [38, 48, 49] since it is not easy to construct a network with given values on the boundary. We also apply penalty method to DRM with ReLU^2 activation functions, and obtain the error estimation in this work. The main contribution are listed as follows: