Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **15**, No. 2, pp. 450-467

DOI: 10.4208/aamm.OA-2021-0257
April 2023

Discontinuous Galerkin Methods for Semilinear Elliptic Boundary Value Problem

Jiajun Zhan[1,2], Liuqiang Zhong[1] and Jie Peng[1,*]

[1] *School of Mathematical Sciences, South China Normal University, Guangzhou, Guangdong 510631, China*
[2] *School of Mathematics and Systems Science, Guangdong Polytechnic Normal University, Guangzhou, Guangdong 510665, China*

Received 5 August 2021; Accepted (in revised version) 10 December 2021

Abstract. A discontinuous Galerkin (DG) scheme for solving semilinear elliptic problem is developed and analyzed in this paper. The DG finite element discretization is first established, then the corresponding well-posedness is provided by using Brouwer's fixed point method. Some optimal priori error estimates under both DG norm and L^2 norm are presented, respectively. Numerical results are given to illustrate the efficiency of the proposed approach.

AMS subject classifications: 65N30, 35J60, 65M12

Key words: Semilinear elliptic problem, discontinuous Galerkin method, error estimates.

---

1 Introduction

Given a bounded domain $\Omega\subset\mathbb{R}^d$ $(d=2,3)$ with the boundary $\partial\Omega$, we assume that $\partial\Omega$ is either smooth or convex and piecewise smooth. We consider the following semilinear elliptic boundary value problem

$$\begin{cases} -\Delta u = f(x,u) & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

For simplicity, we will replace $f(x,u)$ with $f(u)$ in the following exposition. The semilinear elliptic boundary value problem (1.1) is widely used in many practical applications, such as it can be described the potential of a stable fluid or the stable temperature field with a source (see [25]).

*Corresponding author.
Emails: zhan@m.scnu.edu.cn (J. Zhan), zhong@scnu.edu.cn (L. Zhong), pengjie18@m.scnu.edu.cn (J. Peng)

Discontinuous Galerkin (DG) methods are widely used numerical methodologies for solving partial differential equations. They have many advantages in contrast to the conforming finite element methods, e.g., DG methods allow more flexibility in handling equations whose types change within the computational domain and the corresponding finite element space has no continuity constraints across the edges/faces of the triangulation. The DG method was originally used to solve first-order hyperbolic problems [18], and later it was extended to solve various model problems, such as elliptic problems [2], nonlinear elliptic problems [13], Navier-Stokes equations [4], Maxwell equations [9] and so on. The common DG methods usually can be divided into the symmetric interior penalty discontinuous Galerkin (SIPDG) method [21], incomplete interior penalty discontinuous Galerkin method [20], local DG method [11] and so on.

For elliptic problems with nonlinear terms $f(x,u)$, there are only few literatures about DG discretization. Gudi, Nataraj and Pani [12] designed hp-DG methods for strongly nonlinear elliptic boundary value problems, and provided a prior error estimates under DG norm and L^2 norm. Bi, Wang and Lin [5] gave a pointwise error estimates on DG methods for strongly nonlinear elliptic boundary value problems. Houston and Wihler [14] proposed an hp-adaptive procedure for second-order semilinear elliptic boundary value problems. Yadav and Pani [24] provided expanded DG methods, and showed its a priori error estimates and super-convergence phenomenon. Although our model problem is a special case of [12], and we follow the state-of-the-art Brouwer's fixed point theory [12] to prove the corresponding well-posedness of the DG finite element discretization. However, we stress that the extension is not straightforward. Compared with the existing work [12], our contributions in this paper include: our proof process is more concise, and the lower requirements for the regularity is requested (see Remark 2.3), since we introduce another simple non-empty compact convex subset B_h for Brouwer fixed point theorem. Moreover, our results hold for 3D.

In this paper, the semilinear elliptic boundary value problem is solved by SIPDG method. First of all, the DG scheme of problem (1.1) is given, the existence and uniqueness of the finite element solution of the DG scheme is derived by making use of Brouwer fixed point theorem. Then, the optimal priori error estimates in both L^2 norm and DG norm are proved. Finally, numerical experiments are shown to verify the theoretical findings.

To avoid the repeated use of generic but unspecified constants, we shall use $x \lesssim y$ to denote $x \leq Cy$, where constant C is a positive constant independent of the variables that appear in the inequalities and especially the mesh parameters. The notation C_i, with subscript, denotes specific important constant.

The rest of the paper is organized as follows. In Section 2, the DG method is introduced for solving the semilinear elliptic boundary value problem (1.1), and it is proved that the discrete system has unique solution. In Section 3, some optimal prior error estimates under DG norm and L^2 norm are proved, respectively. In Section 4, the numerical experiments are presented. Finally, the main results of this paper are summarized and some concluding remarks are presented in Section 5.