East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.070322.210722

Vol. **13**, No. 1, pp. 140-161
February 2023

A Priori Error Analysis of Mixed Virtual Element Methods for Optimal Control Problems Governed by Darcy Equation

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Received 7 March 2022; Accepted (in revised version) 21 July 2022.

Abstract. A mixed virtual element discretization of optimal control problems governed by the Darcy equation with pointwise control constraint is investigated. A discrete scheme uses virtual element approximations of the state equation and a variational discretization of the control variable. A discrete first-order optimality system is obtained by the first-discretize-then-optimize approach. A priori error estimates of the state, adjoint, and control variables are derived. Numerical experiments confirm the theoretical results.

AMS subject classifications: 65N30, 49J20

Key words: A priori error estimate, mixed virtual element method, optimal control problem.

1. Introduction

The aim of this work is to develop mixed virtual element approximations of optimal control problems governed by the Darcy equation. Let $\Omega \subset \mathbb{R}^2$ be a bounded convex polygonal domain with the Lipschitz boundary Γ. Besides, we also assume that Γ admits a disjoint partition $\Gamma = \overline{\Gamma_1} \cup \overline{\Gamma_2}$ with open subsets Γ_1 and Γ_2 such that $|\Gamma_1|, |\Gamma_2| \neq 0$.

We consider an optimal control problem of the Darcy flow in a porous medium — viz.

$$\min_{u\in U_{ad}} J(\boldsymbol{p}, y, u) = \frac{1}{2}\int_\Omega (\boldsymbol{p}-\boldsymbol{p}_d)^2\,d\Omega + \frac{1}{2}\int_\Omega (y-y_d)^2\,d\Omega + \frac{\gamma}{2}\int_\Omega u^2\,d\Omega \tag{1.1}$$

subject to the following conditions:

$$\text{div}\boldsymbol{p} = f + u \quad \text{in } \Omega, \tag{1.2a}$$

$$\boldsymbol{p} = -\mathbb{K}\nabla y \quad \text{in } \Omega, \tag{1.2b}$$

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$$y = 0 \quad \text{on } \Gamma_1, \tag{1.2c}$$

$$\boldsymbol{p} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_2, \tag{1.2d}$$

where $\boldsymbol{p}$ is the velocity, y the pressure, $J(\boldsymbol{p}, y, u)$ a convex cost functional, $\mathbb{K}$ the permeability of the medium, $\gamma > 0$ a regularization parameter, and $f \in L^2(\Omega)$ the source term. For the sake of simplicity, we assume that $\mathbb{K}$ is a constant matrix and let $\|\mathbb{K}^{-1}\|$ be the matrix norm of $\mathbb{K}^{-1}$. The ideal states $\boldsymbol{p}_d$ and y_d respectively belong to spaces $\boldsymbol{L}^2(\Omega) := (L^2(\Omega))^2$ and $L^2(\Omega)$, and the admissible control set U_{ad} is defined by

$$U_{ad} = \left\{ u \in L^2(\Omega) : a \leq u(x) \leq b \text{ a.e. in } \Omega \right\}.$$

PDEs-constrained optimal control problems play increasingly important role in physics, biology, medicine, and efficient numerical methods are the key to their successful applications. We note that there are numerous studies devoted to the development of numerical methods and algorithms for such problems. In particular, the works [10–12, 16] deal with finite element methods (FEMs) and [15, 21] with discontinuous Galerkin methods. In various practical applications the cost functional often contains the gradient of the state variable. Therefore, the accuracy of the gradient approximation of the state variable becomes an important issue. This enhances interest to mixed numerical methods, such as the mixed finite element method [6]. In optimal control problems, mixed numerical methods have been studied in various works. Thus for mixed finite element discretizations, a priori and a posteriori error estimates of elliptic distributed optimal control problems are investigated [7, 8], whereas the Dirichlet boundary optimal control problem and pointwise state constrained optimal control problem are discussed in [13, 14].

Recently, virtual element methods (VEMs) have attracted a considerable attention. Compared with FEMs, the VEMs can handle general polygonal and polyhedral grids. Such methods have been introduced in [1] in order to solve the elliptic problems. Mixed VEMs have been considered [5], where VEM discretizations of $H(\text{div})$-conforming vector fields are introduced. Subsequently, mixed VEMs have been applied to general linear second-order elliptic problems [2], Darcy and Brinkman equations [18], three-dimensional elliptic equations of mixed formulation [9], the Laplacian eigenvalue problem [17], and optimal control problem governed by elliptic equations [4, 19, 20].

In this paper, we apply a mixed virtual element method to the optimal control problem for the Darcy equation with pointwise control constraints. The state equation is approximated by a mixed virtual element method and the control variable is implicitly discretized. In order to guarantee the computability of the discrete scheme, a piecewise L^2 projection on the discrete state $\boldsymbol{p}_h$ is used in the cost functional. A discrete first-order optimality system is derived by using a first-discretize-then-optimize approach. A priori error estimates for state, adjoint state and control variables are deduced. Two numerical examples are given to illustrate the theoretical results.

The article is organized as follows. In Section 2, we recall auxiliary results related to mixed virtual element methods and the continuous first-order optimality condition in the optimal control problem for the Darcy equation. In Section 3, we consider a virtual